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THE
HISTORY
OF
FLUXIONS,

Shewing in a

COMPENDIOUS MANNER

The firſt Riſe of, and various Improvements made in that Incomparable Method.

By (*the late*) Mr. JOSEPH RAPHSON, *A. M. & R. S. S.*

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G.R. III.
BRITISH MUSEUM

TO THE

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THE

Royal Society

OF

LONDON,

FOUNDED BY

K. CHARLES II.

For the Improvement of

PHILOSOPHY,

And now flouriſhing under the Moſt Auſpicious Government of His Sacred Majeſty King *GEORGE*.

This Treatiſe is moſt humbly dedicated by the BOOKSELLER.

THE
Author's Preface.

IN the writing of this Hiſtory, there were theſe two Things which I propos'd to my Self.

1. *To aſſert the Principal Inventions of this Method, to their* Firſt and Genuine Authors; *and eſpecially thoſe of Sir* Iſaac Newton, *who has vaſtly the Advantage of all others, as well in reſpect of* Priority *of* Time, *as the Great and* Noble Nature *of his* Diſcoveries.

2. *To open a plain and eaſy way for* Beginners *to underſtand theſe* Matters, *and to furniſh them with ſomething that might ſerve as an* Introduction *to the Writings of Sir* Iſaac Newton, *Dr.* Cheyne, *and others, on this moſt Curious Subject: And if this Treatiſe may be of any uſe for that purpoſe, I have obtain'd my End.*

While this was in the Preſs, *there came forth a Book, by Order of the* Royal Society, *with the Title of* Commercium Epiſtolicum, *Containing what lies ſcatter'd up and down in Dr.* Wallis's *Works, and various other Letters never before Publiſh'd, between* Mr. Collins, *Sir* Iſaac Newton, Mr. Leibnitz, *and others. In which the Diſpute between Sir* Iſaac Newton *and* Mr. Leibnitz, *about the* Firſt Invention *of* Fluxions, *is ſet in as clear a Light as can be deſired or hoped for.*

The laſt Chapter *of this Treatiſe, Contains ſome things taken out of that Piece, it being a very ſcarce Book, but few Printed, and none to be met with amongſt the Bookſellers.*

I ſhall detain the Reader no longer, but leave this little Work to his Opinion and Cenſure.

J. R.

CHAP. I.

Of the firſt Author, or Authors of Fluxions.

AFTER the ſeveral Methods of *Exhauſtions* and *Indiviſibles*, firſt of all invented by *Archimedes*, and ſeveral Ages after reviv'd and improv'd (under the latter Name) by *Cavallerius*, and again further improv'd and exhibited in a new Form by the celebrated Dr. *Wallis*, under the Title of *Arithmetick of Infinites*; and after thoſe known Methods of *Mercator* and *Gregory*, and Dr. *Barrow*, from the latter of which (as Mr. *Craig* juſtly complains) *Tſchirnhaus* ſtole moſt Part of what he calls his Inventions, it is evident Sir *Iſaac Newton* about the Years 1665. and 1666. began to advance thoſe Methods (to a very great Perfection) in his own incomparable one of *Fluxions*; which he (by ſeveral Letters) communicated to ſome of the beſt Mathematicians in ſeveral Parts of *Europe*, but which he would never be pleas'd to publiſh till the Year 1704.

About the Year 1670. Mr. *Collins* writ to Mr. *James Gregory*, that Sir *Iſaac* had by him a general Method of Quadratures, as the late Dr. *Gregory* tells us, in his *Geometrical Exercitation*, publiſhed in *Latin* in the Year 1684. *pag.* 3. and among other Mathematicians he communicated the ſame in a Letter in the Year 1676. to Mr. *Leibnitz* (of *Germany*) in (*a*) Letters tranſpos'd as underneath, which Mr. *Leibnitz* having decypher'd, found this Sentence contain'd under them, *viz. Data æquatione fluentes quotcunque quantitates involvente invenire Fluxiones & vice verſa*: In *Engliſh* thus, *Having any given Æquation involving never ſo many flowing Quantities to find the Fluxions and vice verſa*; and ſome time after he perceiv'd his meaning in it, and writ in anſwer that he had found out a Method not unlike it, as Sir *Iſaac* himſelf has hinted, pag. 253. *Princip. Phil. Nat. Math.* but deferr'd publiſhing any thing of his ſaid *own* Invention, till the Month of *October*, in the Year 1684.

(a) *6accdæ13eſſ7i3l9n4o49rr4s9t12vx.*

(*a*) In the ſame Year, *viz.* 1676. Sir *Iſaac*, in a Letter written to Mr. *Oldenburgh* then Secretary to the *Royal Society*, ſignify'd (b) *that he had a Method of drawing of Tangents, after the manner of* Sluſius, *but which was more general than that of* Sluſius; *that his Method did not ſtick at Æquations however adfected by radical Signs involving one or both of the Fluents* (viz. *the Ordinate and the abſciſſe) but that, by it, he could immediately draw the Tangent without any (previous) Reduction of ſuch Æquations, which commonly embarras'd the Operation with almoſt immenſe Trouble: And that this Method would alſo hold not only, for drawing Tangents to Curves, and for determining the* Maxima *and* Minima; *but alſo for ſquaring Curves, and ſolving of other Queſtions, and that by its Help he had found ſome general Theorems for the Quadratures of Curves* (c); and there he ſubjoyns an univerſal Theorem for the Quadrature of *Binomial* Curves, which he calls the *firſt* (viz. of its Rank or kind) that is, the firſt of a Series of Theorems which (after a *Binomial*) may be carried on to a *Trinomial*, *Quadrinomial*, &c. *ad Infinitum*, or to an *Infinitinomial*; and then again, he ſhews the firſt Steps of a Series or Progreſſion of ſuch infinitinomial Series multiplyed into one another, even *ad infinitum*, as ſhall be further explain'd in its proper Place.

He had alſo then (*viz.* in 1676.) contriv'd Methods for extracting the Roots of *Fluxional Æquations*, that is, (to uſe Dr. *Cheyne*'s Words) for finding the *Fluents of ſuch Æquations wherein the Indeterminate (or flowing) Quantities and their Fluxions are ſo confounded together, that keeping the ſame Indeterminate Quantities* (or Letters that repreſent the Fluents) *they cannot eaſily be ſeparated.*

Laſtly, He had then found out general Methods for comparing of ſuch Curves, which could not be Geometrically ſquared, with other more ſimple ones, and had communicated them to ſome of his Acquaintance, as is evident from the (*d*) aforeſaid 2d. Vol. of Dr. *Wallis*'s Works, as alſo from his Treatiſe of Quadratures. So that I am at ſome Loſs to know whether I may ſay, that in theſe things he has anticipated the Inventions of every Body elſe, or given them an Opportunity, by his very free way of communicating his own to his Acquaintance, to publiſh them under other Shapes, or in another Dreſs.

But at Length in the Year 1704. he was pleas'd to publiſh (at leaſt Part of) that Treatiſe reviv'd, which in the Year 1671. he had prepar'd for the Preſs as we mention'd before; and which in 1691. Mr. Profeſſor *Halley*, and I, had in our Hands at *Cambridge*, in order to bring it

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(a) *See* Wall. *Vol.* II. *Op.* Math. *p.* 390.

(b) *A Treatiſe he had formerly written of this Nature, had the Misfortune to be burnt, as is hinted in Dr.* Wallis, Vol. II. p. 390. *Vid.* p. 391.

(c) *See* p. 393. *and at the End of one of his* Lectures, *in the Year* 1676.

(d) *Page* 393. *as above.*

up to be printed, and which was then very much worn by having been lent out; but he at that time deſir'd it back again for ſome further Reviſal, and never could be perſwaded to print it afterwards, till the ſaid Year, 1704.

Sir *Iſaac* therefore, before or about the Years *1665.* and *1666.* found out this Method, and in the Year *1676.* gave ſome Notice of it to (*a*) Mr. *Leibnitz*, who a conſiderable time afterwards, *viz.* in the Month of *October*, 1684. firſt publiſh'd the Principles of his *Differential Calculus* in the *Act. Erud.* that is nineteen Years after the ſame was found out by Sir *Iſaac*, and 8 Years after Sir *Iſaac* had communicated theſe Notices of it to him.

There have not been wanting ſome that have imagin'd that Dr. *Barrow* gave the firſt Hints to this Method; among the reſt *J. Bernoulli* was of this Opinion in the *Act. Eruditor.* for the Month of *January*, in the Year 1691, as appears by theſe Words of his; *viz.* Quanquam, *&c.* *Altho' to confeſs the Truth, he that underſtands Dr.* Barrow's *Calculus, which ten Years ago he read in his Geometrical Lectures,* (*viz.* in the Univerſity of *Cambridge*) *and whereof all the Propoſitions of that Book are a Specimen, cannot be ignorant of the lately invented* Calculus *of Mr.* Leibnitz, *becauſe the former is founded upon it, and no way differs from it, except perhaps in the Notation of the Differential Quantities and ſome Compendiouſneſs in the Operation.* But afterwards (*viz.* in the Month of *June*, in the ſame Year, he began to think he had made too large Conceſſions, and therefore by way of Retractation talks after this Rate of his (*ſuppoſed*) *Leibnitian* Invention; *Cæterum in hiſce*, &c. *But in all theſe Problems which are attempted in vain by any other Method, I have always found the extraordinary Uſe and Advantage of the* Leibnitian Calculus, *ſo that I can't help ranking it amongſt the firſt and moſt extraordinary Inventions of this Age. For altho', as I have not long ago mention'd, I believe Dr.* Barrow's Calculus *(as I have call'd it) gave the firſt Hints, and Riſe to this, and which from his Time has almoſt univerſally obtain'd among the beſt* Geometricians, *and which I ſee Mr.* Tſchirnhaus *always makes uſe of: Yet I don't mean it ſo, as if I any way deſign'd either to elevate the Excellency of this noble Invention, or to detract from the Honour that is juſtly due to its Author, much leſs to adſcribe it to others; and if while I was comparing them together, there ſometimes ſeem'd to me to be ſome Affinity between them, it was only in this particular, that to a Perſon who underſtood the one, the whole Reaſon of the other would preſently become apparent; while one only ſtrikes out the ſuperfluous Quantities, which afterwards muſt have been ſtruck out; and the other more compendiouſly omitts them. As for the reſt the Compendium (or Shortneſs of the Management) is ſuch that quite changes*

(a) *See* Phil. Nat. Princip. Math. *p.* 252.

changes the Nature of the thing, and makes an Infinity of Caſes ſolvible by this, which are not ſo by the other: And as the Invention alone of ſuch a Compendium *is what was not deſign'd for the common Rank of Mathematicians, but ſor ſome extraordinary Genius, ſo it will always redound to the Honour of its Author.*

But that the whole Buſineſs may be ſet in its true Light, I have deſign'd in the following Papers (by way of *Synopſis*) to lay open the chief Inventions (as far as I can trace them) of the firſt Authors, and that in their own Words, adding for the Sake of thoſe that may want ſuch a Help, an *analytical* Explication of their Methods, in order to acquaint the Reader with the Inveſtigation of them, to make the *Praxis* of the whole as eaſy as poſſible, but chiefly to do Juſtice to their Authors in Point of *Chronology*, which was the chief Reaſon I gave it the Title of an *Hiſtory*.

CHAP. II.

Of NOTATION, &c.

WHAT Sort of *Notation* Sir *Iſaac* made uſe of at the Beginning of this his Invention does not publickly appear; tho' I imagine he might at firſt fall upon the ſame Method of Punctation that he uſes afterwards. In a Lemma in *Pag.* 256. of his *Princip. Phil. Nat. Math.* where he gives the Algorithm of this Method, he uſes another in the Demonſtration of it, denoting the *Fluents* in Capital Letters, and their *Fluxions* in ſmall ones, as A, B, C, *&c.* a, b, c, *&c.* reſpectively. *(a)* Mr. *Nieuwentijt*, in denoting the Differentials of his indeterminate Quantities (x, y, &c.) follows Dr. *Barrow* in aſſigning the Vowels a, e, i, o, u, for their Differentials (or *Fluxions*) which he calls *infiniteſimal Parts*.

Mr. *Leibnitz* choſe another Way of Notation, different from both theſe, wherein he is followed by the two Brothers, *John* and *James Bernoulli*, and by the *German* and *French* Nations and other Parts of *Europe*. He expreſſes the *Differences* (or *Fluxions*) of his indeterminate Quantities, *viz.* x, y, z, &c. by a diſtinguiſhing d, viz. dx, dy, dz, &c. the Second Difference thus, ddx, ddy, ddz, &c. the third, $dddx$, $dddy$, $dddz$,

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(*a*) Analyſis Infinitorum.

dddz; as alſo $\overline{dx}|^3$, $\overline{dy}|^3$, $\overline{dz}|^3$, and ſo on: This way of Notation is follow'd among us by Mr. *John Craig*. But the firſt and moſt learned Author of theſe Methods, Sir *Iſaac Newton*, makes uſe of Points, and denotes thoſe firſt *Differences* (which by a Name congruous to their Generation, being conſider'd as the firſt Increments or Decrements of a continued Motion, he calls *Fluxions*) thus, *viz.* $\dot{x}$, $\dot{y}$, $\dot{z}$; and their ſecond *Fluxions* by double Punctations thus, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$; and the third thus, $\dddot{x}$, $\dddot{y}$, $\dddot{z}$, *&c. ad infinitum.* To theſe Quantities he adds others of another Gender, and which in relation to Finite ones may be conceiv'd as infinitely great, and denotes them thus $\acute{x}$, $\acute{y}$, $\acute{z}$, whereof the firſt or finite Quantities themſelves, *viz.* x, y, z, may be conceiv'd as *Fluxions*, and they themſelves as Fluxions of others of a yet higher Gender or ſecond Degree of Infinity, *viz.* of $\overset{\prime\prime}{x}$, $\overset{\prime\prime}{y}$, $\overset{\prime\prime}{z}$; and thoſe again of others, as $\overset{\prime\prime\prime}{x}$, $\overset{\prime\prime\prime}{y}$, $\overset{\prime\prime\prime}{z}$, *&c. ad infinitum*; ſo that they would conſtitute a Series after this manner, *viz.* $\overset{\prime\prime\prime\prime}{z}$, $\overset{\prime\prime\prime}{z}$, $\overset{\prime\prime}{z}$, $\overset{\prime}{z}$, z, $\dot{z}$, $\ddot{z}$, $\dddot{z}$, $\ddddot{z}$, &c. infinitely continuable on both ſides of the ordinary (or finite) Quantity, wherein every ſubſequent one may be conſider'd as a Fluxion of the precedent, and conſequently every precedent one as a Fluent of the ſubſequent. After the ſame manner the following Series's are to be underſtood, *viz.* $\sqrt{az\overset{\prime\prime}{-}zz}$, $\sqrt{az\overset{\prime}{-}zz}$, $\sqrt{az-zz}$, $\sqrt{az\overset{.}{-}zz}$, $\sqrt{az\overset{..}{-}zz}$, $\sqrt{az\overset{\therefore}{-}zz}$, &c. as alſo this $\overset{\prime\prime}{\frac{az+zz}{a-z}}$, $\overset{\prime}{\frac{az+zz}{a-z}}$, $\frac{az+zz}{a-z}$, $\overset{.}{\frac{az+zz}{a-z}}$, $\overset{..}{\frac{az+zz}{a-z}}$, *&c. ad inf.* where (ſays Sir *Iſaac*) *every former Quantity is the Area of a Curve, whoſe rectangular Ordinate is the next ſubſequent Quantity, and its Abſciſſe* z; *as* $\sqrt{az\overset{\prime}{-}zz}$ *will be the Area of a Curve whoſe Ordinate is* $\sqrt{az-zz}$ *and its Abſciſſe* z. Thus a Point (taken in a leſs reſtrain'd ſenſe) may be conſider'd as the Fluxion of a Line, a Line as the Fluxion of a Plane, and a Plane as the Fluxion of a Solid, and a finite Solid as the Fluxion of a (partially) infinite one, and that again as the Fluxion of one of an higher Gender of Infinity, and ſo on *ad inf.* which we ſhall further illuſtrate in ſome Diſſertations at the end of this Treatiſe. Thus alſo ſuch Series as theſe, *viz.*

$$\frac{\dot{x}}{x},\ \frac{\dot{x}}{x'},\ \frac{\dot{x}}{x''},\ \frac{\dot{x}}{x'''},\ \text{\&c.}\ \ \frac{\ddot{x}}{x'},\ \frac{\ddot{x}}{x''},\ \frac{\ddot{x}}{x'''},\ \text{\&c.}\ \ \frac{\dddot{x}}{x'},\ \frac{\dddot{x}}{x''},\ \frac{\dddot{x}}{x'''},\ \text{\&c. ad Inf.}$$

as alſo ſeveral others, may be made uſe of to repreſent Quantities of different Genders upon different occaſions: And if any Reader is ſcrupulous of admitting theſe and the like Degrees of Fluxions, they may pleaſe to conſider that Sir *Iſaac Newton* and Mr. *Leibnitz* (tho' in ſomewhat a different ſenſe from one another) have long ago admitted them, Mr. *Leibnitz*, in the *Act Erud.* for the Month of *February* 1689, in an Eſſay about the *Cauſes of the Cæleſtial Motions*, writes thus, *Aſſumſi inter demonſtrandum*, &c. *I have made uſe in my Demonſtrations of Quantities incomparably ſmall*, viz. *of the difference of two common or ordinary Quantities, incomparable (by reaſon of its ſmallneſs) to the Quantities themſelves. And thus, if I am not miſtaken, theſe Quantities may eaſily become intelligible; and if any one refuſes to admit of Quantities that are infinitely ſmall, he may aſſume them as ſmall as he thinks will ſuffice for his purpoſe, that they may be incomparable, and produce an Error of no moment, or leſs than any given (or aſſignable) one. As* e. g. *the Earth (itſelf) may be reckon'd as a Point, or the Diameter of the Earth as a Line, infinitely ſhort, in compariſon to the Heaven. Thus it may be demonſtrated, if the Sides of an Angle ſtand on ſo ſmall a Baſe, that it bears no Compariſon to them, that the Angle comprehended under thoſe Sides will be* incomparably *leſs than a right one; and that the difference of the Sides,* (upon the ſuppoſition there is any) *will be incomparable to the Sides themſelves; alſo that the difference of the whole Sine, Sine Complement, and Secant, will be incomparable to the Quantities differing, as will be likewiſe the difference of the Sine, Chord, Arch, and Tangent; whence when theſe themſelves are infinitely ſmall, their differences will be infinitely infinite ſmall, and the verſed Sine will alſo be infinitely infinite ſmall, and conſequently incomparable to the right Sine: and there are infinite Degrees as well of infinite* [Quantities,] *as of infinitely ſmall ones. And you may make uſe of common Triangles ſimilar to thoſe unaſſignable* [or infinitely ſmall ones,] *which are of great uſe in drawing Tangents, and in determining the* Maxima *and* Minima, *and explaining the Curvature of Lines, and almoſt univerſally in the* Tranſlation [*or* Application] of Geometry to Nature; *for if Motion be expreſs'd by a common Line which the moveable Body abſolves in any given time, the* Impetus *or Velocity will be expreſs'd by a Line infinitely ſmall, and the very firſt* [Beginning *or*] *Element of the Velocity, ſuch as is the centripetal Tendency of heavy Bodies, or the* Conatus centrifugus, *by a Line infinitely infinite ſmall*, &c. Sir *Iſaac Newton* had long before talk'd much after the ſame manner, but infinitely more cautiouſly, as well as naturally, as may be ſeen in *pag.* 165 of his *Opticks*: *I here conſider Mathematical Quantities not as conſiſting of* [infinitely] *ſmall Parts, but as deſcrib'd by continual Motion: Lines are deſcrib'd, and generated by being ſo deſcrib'd, not by the Appoſition of Parts, but by the continual Motion of Points, Surfaces by the Motion of Lines, Solids by the Motion of Surfaces, Angles by the Rotation* [or Circulation] *of their Sides, Time by a continued Flux, and ſo in other things. The Gene-*

rations of theſe Quantities are really in Nature, and are daily ſeen in the Motion of Bodies. But to return to Notation: Sir *Iſaac* brought into uſe ſome others, (occaſion obliging him *to change ſome of the former ways*, as he himſelf acquaints us) more commodiouſly to perform the Operations of this *Calculus*, and which now have been long in uſe among thoſe Mathematicians that have follow'd him; which may be ſeen in Dr. *Wallis*, p. 315. Vol. 3. of the laſt Edition; as ſubſtituting

$$
\begin{array}{ll}
\textit{for}\ \frac{1}{x} \cdots\cdots & x^{-1} \\
\frac{1}{xx} \cdots\cdots & x^{-2} \\
\frac{1}{x^3} \cdots\cdots & x^{-3}\ \text{\&c.} \\
\sqrt{x} \cdots\cdots & x^{\frac{1}{2}} \\
\sqrt{x^3} \cdots\cdots & x^{\frac{3}{2}} \\
\sqrt[3]{x} \cdots\cdots & x^{\frac{1}{3}} \\
\frac{1}{\sqrt{x}} \cdots\cdots & x^{-\frac{1}{2}} \\
\frac{1}{x^{\nu}} \cdots\cdots & x^{-\nu} \\
\frac{a}{x} \cdots\cdots & ax^{-1}
\end{array}
$$

$$
\begin{array}{ll}
\textit{and for the Square of}\ \overline{e+fx},| \cdots & \overline{e+fx}|^{2} \\
\sqrt{aa-xx} \cdots & \overline{aa-xx}|^{\frac{1}{2}} \\
\frac{1}{\sqrt{aa-xx}} \cdots & \overline{aa-xx}|^{-\frac{1}{2}} \\
\frac{dx}{\overline{e+fx}|^{2}} \cdots & dx \times \overline{e+fx}|^{-2}\ \text{\&c.}
\end{array}
$$

To theſe may be added alſo thoſe Operations of his in relation to the *Indices* of Powers, as,

1. That the Addition of the Indices is equivalent to the Multiplication of the Powers; as $x^2 \times x^3 = x^{2+3} = x^5$.

2. The Subſtraction of Indices is equivalent in this Method of Expreſſion of Indices, to the Diviſion of Powers; thus $\frac{x^3}{x^2} = x^{3-2} = x^1 = x$.

3. The Multiplication of Indices is equivalent to the Involution of the Powers to which they are prefix'd; thus $x^{3\times 3}$ *is* $=$ *Cub.* $x^3 = x^9$.

4. The Diviſion of the Indices is equivalent to Evolution, or the Extraction of Roots; *e. g.* $x^{\frac{6}{3}} = \sqrt[3]{x^6}$ becauſe $x^{2\times 3} =$ to the Cube of $x^2 = x^6$.

Sir *Iſaac* moreover has very commodiouſly brought into uſe another Method of Notation, in his above-mention'd Treatiſe of Quadratures, particularly of a very extraordinary and commodious uſe in theſe caſes; *viz.* by ſubſtituting a Capital Letter for a whole Series. See page 176 of his *Quadratures.* For example, make

$$e+fz^{\eta}+gz^{2\eta}+hz^{3\eta}, \text{\&c.} = R.$$
$$k+lz^{\eta}+mz^{2\eta}+nz^{3\eta}, \text{\&c.} = S, \text{\&c. \&c.}$$

Then will $\overline{e+fz^{\eta}+gz^{2\eta} \text{ \&c.}}|^{\lambda} = R^{\lambda}$ &c. &c. whence ſubſtituting $RR^{\lambda-1}=R^{\lambda}$, or for R^{λ}, both in the Operation and the Demonſtration, but more particularly as a directory Canon for ordering the Series, of what uſe it will be, ſhall be hereafter ſhewn. I ſhall not here trouble my Reader with the particular Notations of ſome other Perſons, only made uſe of perhaps to prevent (if it had been poſſible) the Suſpicion of their having been *Plagiaries*, by differing from ſome others that had been communicated to them before (ſuch are ſome of *Leibnitz*) as an ingenuous Reader of both will eaſily gueſs, becauſe it will not be worth my while to trouble him with 'em here.

CAP. III.

Of ſpecious Involution & Evolution.

AMong the reſt of Sir *Iſaac*'s Inventions (which are of uſe particularly to the Method of Fluxions) that is none of the leaſt which he firſt of all invented, for raiſing any ſpecious Quantity to any Power deſir'd, or for extracting any given Root of it.

He imparted the ſame (as appears in the 91ſt Chapter of the ſecond Volume of Dr. *Wallis*) in the Year 1676 in two Letters, to Mr. *Oldenburgh*, then Secretary of the *Royal Society*, to be communicated to the Society, which Letters are extant in the third Volume of Dr. *Wallis*'s Works, and mention made of them in the ſecond Volume, page 369. There he ſhews that the *Unciæ, or Numbers prefix'd to the reſpective Members of Powers rais'd from Binomial Roots, the general Exponent of all which, (to be particularly determin'd as occaſion requires) be there denotes by the Letter m, ariſe by a continued Multiplication of this Series,* viz.

$$1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \text{ \&c.}$$

Which Proceſs, if m (the Exponent of the Power) be an Integer Number, will be determin'd after a certain Number of Places as each Power ſhall reſpectively require, ending with Unity as it begun. But if m (the Exponent of the Power) be a Fraction, the Series will continue on ad infinitum *in negative Members.* But as this Theorem is now grown common among Mathematicians, we ſhall make no farther mention of it. From this Proceſs (and ſome other of the ſame nature) he deduces theſe and ſuch like Theorems for the raiſing of Powers and Extraction of Roots.

1°. $\overline{a+x}|^{m} = a^{m} + mAQ + \frac{m-1}{2}BQ + \frac{m-2}{3}CQ + \frac{m-3}{4}DQ$, &c. or

2°. $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ$ &c.

Where $P+PQ$ is the given Quantity the Root whereof is to be extracted, or ſome Power to be rais'd from it: P is the firſt Term of the ſaid Quantity, Q is its Remainder (whether that Remainder be a *Binome* or a *Trinome*, &c.) divided by P; and $\frac{m}{n}$ the Exponent of the Root or Dimenſion ſought. This Remainder is ſo to be underſtood (as may be ſeen *pag.* 376.) as that it may ſerve both for raiſing of Powers, and extracting of Roots of a Quantity of ſeveral Members, which will either run out into an infinite Series, or a finite Quantity, according as the caſe requires. He has added eight Examples of ſuch a Proceſs (*Chap.* 91.) whereof we ſhall here tranſcribe only two or three.

Examp. 1.

“ $\sqrt{cc+xx}$ or $\overline{cc+xx}|^{\frac{1}{2}} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{x^8}{128c^7} + \frac{7x^{10}}{256c^9}$ &c.

“ For in this caſe you'll have $P = cc$. $Q = \frac{xx}{cc}$. $m = 1$. $n = 2$. $A = (P^{\frac{m}{n}} =$
“ $cc^{\frac{1}{2}}) = c$. $B (= \frac{m}{n} AQ) = \frac{xx}{cc}$. $C (= \frac{m-n}{2n} BQ) = \frac{-x^4}{8c^3}$ &c.

Examp. 2.

“ $\sqrt[5]{c^5 + c^4 - x^5}$ or $\overline{c^5 + c^4 - x^5}|^{\frac{1}{5}} = c + \frac{c^4x - x^5}{5c^4}$
“ $- \frac{2c^8 + 4c^4x^5 - 2x^{10}}{25c^9} +$ &c. Which ſubſtituting $1 = m$,
“ $5 = n$, $c^5 = P$. & c^5) $c^4x - x^5$ (Q. Or alſo ſubſtituting $-x^5 = P$
“ & $-x^5$) $c^4x + c^5$ (Q. $\sqrt[5]{c^5 + c^4x - x^5}$ will be $= -x + \frac{c^4x + c^5}{5x^4}$

“ $+ \frac{2c^8x^2 + 4c^9x + c^{10}}{25x^9} +$ &c. The firſt is moſt eligible if x be very
“ ſmall, the latter if it be very great.

Examp. 3.

“ $\frac{N}{\sqrt[3]{y^3 - aay}}$, that is $N \times \overline{y^3 - aay}|^{-\frac{1}{3}} = N \times \frac{1}{y} + \frac{aa}{3y^3} + \frac{a^4}{3y^5} + \frac{7a^6}{81y^7}$ &c.
“ For here P is $= y^3$. $Q = \frac{-aa}{yy}$. $m = 1$. $n = 3$. $A \left(= \frac{m}{n} AQ = -\frac{1}{3} \times \frac{1}{y}\right.$
“ $\left. \times \frac{-aa}{yy}\right) = \frac{aa}{3y^3}$ &c. &c. &c.
“ And by the ſame Rule we may alſo perform, as well in Numbers as
“ in Species, the Generation of Powers, Diviſion by Powers, or by
“ radical Quantities, and the Extraction of the Roots of ſuperior
“ Powers, and the like. Among other Inveſtigations of theſe Theorems, I ſhall only ſet down the following one here, which Mr. *Heynes* formerly communicated to me, before any other I have ſeen were extant, tho' ſomewhat out of place, *viz.* To inveſtigate any Root or Power of $a + x$, which *e. g.* ſhall be denoted by m, *viz.* $\overline{a+x}|^m$.

Aſſume $\overline{a+x}|^m = b + cx + dx^2 + ex^3$ &c.

Then will $m\overline{a+x}|^{m-1}\dot{x} = c\dot{x} + 2dx\dot{x} + 3ex^2\dot{x}$ &c. And dividing by $\dot{x}$.

$$m \times \overline{a+x}|^{m-1} = c + 2dx + 3ex^2, \text{ \&c.}$$

Then will $\frac{m \times \overline{a+x}|^{m-1}}{\overline{a+x}|^m} = m \times \overline{a+x}|^{-1} = \frac{m}{a+x} =$

$\frac{c + 2dx + 3ex^2 + 4fx^3 \text{ \&c.}}{b + cx + dx^2 + ex^3 \text{ \&c.}}$ & conſequently $bm + cmx + dmx^2 + emx^3$ &c.

$$\begin{array}{l} = ac + 2adx + 3aex^2 + 4afx^3 \text{ \&c.} \\ \qquad\qquad cx + 2dx^2 + 3ex^3 \text{ \&c.} \end{array}$$

and by putting $b = a^m$, as the nature of Powers requires, you'll have

$$\left.\begin{array}{l} c = ma^{m-1} \\ d = \frac{m-1}{2} \times ma^{m-2} \\ e = \frac{m-2}{3} \times \frac{m-1}{2} \times ma^{m-3} \end{array}\right\} \text{\&c.}$$

By comparing the Homologous Terms.

$$\text{\&}\ \overline{a+x}|^m = a^m + ma^{m-1}x + \frac{m-1}{2} \times ma^{m-2}x^2 + \frac{m-2}{3} \times \frac{m-1}{2} \times ma^{m-3}x^3 \text{ \&c.}$$

$$= a^m + mA\frac{x}{a} + \frac{m-1}{2}B\frac{x}{a} + \frac{m-2}{3}C\frac{x}{a} + \text{\&c.}$$

And

Then putting A for the firſt Term, B for the ſecond, C for the third, *&c* *&* *making* $\frac{a}{x}$=Q. *you'll have* $\overline{a+x}|^m = a^m + m\text{AQ} + \frac{m-1}{2}\text{BQ} + \frac{m-2}{3}\text{CQ} +$ *&c.* Which is the very Expreſſion of Sir *Iſaac Newton* we juſt now ſhew'd *&c.* And equalling the Quantities as the Caſe requires, the other comes out after the ſame way. The ingenious Mr. *Abr. de Moivre* likewiſe in the *Philoſoph. Tranſact.* N° 230, ſhews a Method for elevating any multinomial or infinitinomial Quantity to any indeterminate Power, or for extracting any given Root of it, which we ſhall here tranſlate from his own Words. But you are firſt to note that m (the Index) ſtands either for an Integer or Fraction, as the caſe requires.

$$\overline{az+bz^2+cz^3+dz^4+ez^5\ \&c.}|^m = a^m z^m + \frac{m}{1}a^{m-1}bz^{m+1} + \frac{m}{1}\times\frac{m-1}{2}$$

$$\left.\begin{array}{l} a^{m-2}b^2 \\ \frac{m}{1}a^{m-1}c \end{array}\right\} x^{m+2}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}b^3 \\ \frac{m}{1}\times\frac{m-1}{1}a^{m-2}bc \\ \frac{m}{1}a^{m-1}d \end{array}\right\} z^{m+3}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}a^{m-4}b^4 \\ \frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}a^{m-3}b^2c \\ \frac{m}{1}\times\frac{m-1}{1}a^{m-2}bd \\ \frac{m}{1}\times\frac{m-1}{2}a^{m-2}c^2 \\ \frac{m}{1}a^{m-1}e \end{array}\right\} z^{m+4}\ \&c.\ \&c.$$

" In this Series the Law of Continuation is this. To find all the Members of any Term, I multiply the Members of the Term immediately preceding it by $\frac{b}{a}$, alſo all the Members of the Term immediately preceding this laſt I multiply by $\frac{c}{a}$ excepting thoſe wherein is found b. Again I multiply all the Members of the Term in like manner preceding this laſt by $\frac{d}{a}$, except thoſe in which there is found b or c. And all the Members of the Term immediately preceding this laſt I multiply again by $\frac{e}{a}$, excepting thoſe in which is found b, or c, or d. And thus by going back thro' all the Terms before found, you'll find by collecting them all the Members that enter that (reſpective) Term of the Series; whoſe *Unciæ*, or the Num-

" bers to be prefix'd to 'em, are thus to be found, For any Member " of any Term of the Series, take ſo many Terms of this Series " $m \cdot \times m-1 \times m-2 \times m-3$, &c. as there are Units in the Sum of the " * Exponents of the Letters *b*, *c*, *d*, *e*, &c. in that Member; make " this the Numerator of a Fraction, whoſe Denominator will be " the Product of the Series's $1 \times 2 \times 3 \times 4$, &c. $1 \times 2 \times 3 \times 4$, &c. " $1 \times 2 \times 3 \times 4 \times 5$, &c. the firſt whereof has as many Terms as there " are Units in the Exponent of *b*, the ſecond as many Terms as there " are Units in the Exponent of *c*, the third as many as there are U- " nits in the Exponent of *d*, &c. The Demonſtration may be ſeen in the place aforeſaid.

Dr. *Cheyne* has furniſh'd us with a very expeditious Method to the ſame purpoſe, and which (as he acquaints us in the 58th Page of his *Inverſe Meth.)* may be continued *ad inf.* by Inſpection, *viz.*

$$\overline{a+bx+cx^2+dx^3+ex^4\ \&c.}\big|^n = a^n + \frac{nbA}{a}x + \frac{2ncA+\overline{n-1}B}{2a}x^2$$

$$+\frac{3ndA+\overline{2n-1}cB+n-2bC}{3a}x^3 + \frac{4neA+\overline{3n-1}dB+\overline{2n-2}cC+\overline{n-3}bD}{4a}x^4\ \&c.$$

And the Tenth Term of courſe will be

$$\frac{5nkA+\overline{8n-1}jB+\overline{7n-2}bC+\overline{6n-3}gD+\overline{5n-4}fE+\overline{4n-5}eF+\overline{3n-6}dG+\overline{2n-7}cH+n-8bJ}{9a}x^9\ \&c.\ \&c.$$

" In this Series every great Letter denotes the whole Coefficient im- " mediately preceding it, where it firſt ariſes or is brought in, and " every where is of the ſame value. He adds an Example yet more complicated, which the Reader may, if he pleaſes, at his Leiſure conſult. He adds alſo this juſt *Memorandum* to theſe Methods, *viz.* " All that " relates to this Affair may be had from the firſt Inventor Sir *Iſaac New-* " *ton*; for if (as he adviſes) for the firſt Term of the Quantity to be " elevated you put *p*, for the reſt, excepting the firſt, you put *q*, you'll " eaſily obtain whatever can be deſir'd of this nature.

But there is nothing more eaſy than to obtain the Terms of ſuch a Series as far as you pleaſe, by mere Inſpection, if you diſpoſe them after a Tabular Method, thus, *viz.*

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* *Mr.* Moivre *by the Exponent of a Letter means the Number which expreſſes what Place the Letter has in the Alphabet; So* 3 *is the Exponent of the Letter* c, *becauſe the Letter* c *is the third in the Alphabet*, &c.

1	a^n										
2	$nbAx$										
3	$2ncA$	$n-1bBx^2$									
4	$3ndA$	$2n-1cB$	$n-2Cx^3$								
5	$4neA$	$3n-1dB$	$2n-2cC$	$n-3bDx^4$							
6	$5nfA$	$4n-1eB$	$3n-2dC$	$2n-3cD$	$nb-4Ex^5$						
7	$6ngA$	$5n-1fB$	$4n-2eC$	$3n-3dD$	$2n-4cE$	$n-5bFx^6$					
8	$7nhA$	$6n-1gB$	$5n-2fC$	$4n-3eD$	$3n-4dE$	$2n-5cF$	$n-6bGx^7$				
9	$8niA$	$7n-1hB$	$6n-2gC$	$5n-3fD$	$4n-4eE$	$3n-5dF$	$2n-6cG$	$n-7bHx^8$			
10	$9nkA$	$8n-1iB$	$7n-2hC$	$6n-3gD$	$5n-4fE$	$4n-5eF$	$3n-6dG$	$2n-7cH$	$n-8bIx^9$		
11	$10nlA$	$9n-1kB$	$8n-2iC$	$7n-3hD$	$6n-4gE$	$5n-5fF$	$4n-6eG$	$3n-7dH$	$2n-8cI$	$n-9bKx^{10}$	
12	$11nmA$	$10n-1lB$	$9n-2kC$	$8n-3iD$	$7n-4hE$	$6n-5gF$	$5n-6fG$	$4n-7eH$	$3n-8dI$	$2n-9cK$	$n-10bLx^{11}$ &c. ad inf.

And every Member of this Series is divided, *viz.* the firſt by a, the ſecond by $2a$, the third by $3a$, and ſo on *ad inf.* that is, the Series's above make the Numerators, and a, $2a$, $3a$, &c. the Denominators of each Member.

In its proper place we ſhall ſhew the Inveſtigation of this Theorem. p. 64

It is ſufficiently evident that by either of theſe Methods, the Powers and Roots of *Trinomial*, *Quadrinomial* Quantities, *&c.* may be obtain'd, *viz.* by making all the Coefficients in the general Series, after the laſt of any particular *Nome* ſought, equal to nothing. So that nothing farther ſeems to be wanting to complete this Method, that is not as eaſily obtain'd as the Reader can wiſh.

CHAP. IV.

Of the direct Method of Fluxions.

IN the Year 1687, Sir *Iſaac Newton* publiſh'd that incomparable Treatiſe of his, entitl'd *Principia Philoſophiæ Naturalis Mathematica*; which is great part owing to this *Calculus*, as is juſtly obſerv'd by the Marquis *de l' Hoſpital*, in theſe Words in his Preface to his Treatiſe entitl'd *L' Analyſe des Infiniment Petits*, &c. wherein he, as well as ſome other Geometricians of *Europe*, particularly of *Germany* and *France*, having been perſuaded that Mr. *Leibnitz* was the firſt Inventor of this *Calculus*, yet as if he ſomewhat ſuſpected it, he openly breaks out into theſe words, *Il eſt vray*, &c. *It is true the Learned Mr.* Newton *has found out ſomewhat of the like nature, as appears by his excellent Book entitl'd* Principia, *&c. which is almoſt all of this* Calculus. It was in this Treatiſe, I ſay, that Sir *Iſaac* firſt imparted to the Publick the *Algorithm* of this his *Calculus*, pag. 256. And there he ſhews almoſt innumerable Applications of it both to *Geometry* and *Nature*, and had long before communicated ſeveral *Specimens* of it to ſeveral Mathematicians of *Europe*, with whom he kept a Correſpondence: But in this Demonſtration he made uſe of another way of Notation than what he ſince ſhews. And he afterwards communicated the ſame to Dr. *Wallis* in his Method now in uſe, as is evident in pag. 292 of his ſecond Volume, and which he has inſerted into, or rather there tranſcrib'd out of, his Treatiſe of Quadratures, in theſe words, *viz. Having any given Æquation involving never ſo many flowing Quantities, to find their Fluxions.*

SOLUTION.

Multiply every Term of the Æquation by the Index of every flowing Quantity it involves, and in each Multiplication change one of the Sides of each Power into its Fluxion, and the Aggregate or Sum of all the Factors collected under their proper Signs, will exhibit a new Æquation expreſſing (or ſhewing the relation of) the Fluxions.

EXPLICATION.

Let a, b, c, d, *&c. be determinate (or unalterable Quantities) and let there be propos'd an Æquation involving any Number of flowing Quantities,* z, y, x, *&c. as ſuppoſe* $x^3 - xyy + aaz - b^3 = 0$: *in the firſt place multiply the Terms involving* x *by the reſpective Indices of all its Powers, and in each Multiplication for one Side of the Power, or for one ſingle* x *in each write* $\dot{x}$, *and the Sum of the Products will be* $3\dot{x}x^2 - \dot{x}yy$. *Do the ſame for* y, *and you'll have* $2xy\dot{y}$. *Laſtly do the ſame for* z, *and you'll have* $aa\dot{z}$. *Make the Sum of the Products equal to nothing, and you'll have the Æquation* $3\dot{x}x^2 - \dot{x}yy - 2xy\dot{y} + aa\dot{z} = 0$. *I ſay the Relation of the Fluxions will be defin'd by this Æquation.*

DEMONSTRATION.

For ſuppoſe o *to be a Quantity ſufficiently ſmall, and let* $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, *be the Moments of* z, y, x, *that is, the equal momentary Increments of the Quantities* z, y, x. *And now if the flowing Quantities are* z, y, x, *they will* [viz. by an uniform continual Motion] *when augmented by their ſmall Increments* $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, *after a Moment of Time, become* $z + o\dot{z}$, $y + o\dot{y}$, $x + o\dot{x}$, *which being writ in the firſt Æquation for* z, y, x, *produce this Æquation* $x^3 + 3xxo\dot{x} + 3xoo\dot{x}\dot{x} + o^3\dot{x}^3 + xyy - o\dot{x}yy - 2xoy\dot{y} - 2\dot{x}ooy\dot{y} - xoo\dot{y}\dot{y} - xo^3\dot{y}\dot{y} + aax + aao\dot{z} - b^3 = 0$. *Subtract*

Subtract the former Æquation, and the Remainder divided by o *will be*

$$3\dot{x}x^2 + 3\dot{x}\dot{x}ox + \dot{x}^3oo - \dot{x}yy - 2x\dot{y}y - 2\dot{x}o\dot{y}y - xo\dot{y}\dot{y} + aax = o.$$

Now diminiſh the Quantity o in infinitum, *and neglecting the evaneſcent Terms, there will remain* $3\dot{x}x^2 - \dot{x}yy - 2x\dot{y}y + aa\dot{z} = o$. *Q. E. D.*

A more full Explication.

After the ſame way if the Æquation was $x^3 - xyy + aa\sqrt{ax-yy} - b^3 = o$, *there would be produc'd* $3x^2\dot{x} - \dot{x}yy - 2xy\dot{y} + aa\overset{\cdot}{\overline{\sqrt{ax-yy}}} = o$. *Where if you wou'd exterminate the Fluxion* $\overset{\cdot}{\overline{\sqrt{ax-yy}}}$, *make* $\sqrt{ax-yy} = z$, *and* $ax - yy$ *will be* $= z^2$, *and (by this Propoſition)* $a\dot{x} - 2y\dot{y} = 2z\dot{z}$ *or* $\frac{a\dot{x} - 2y\dot{y}}{2z} = \dot{z}$, *that is* $\frac{a\dot{x} - 2y\dot{y}}{2\sqrt{ax-yy}} = \overset{\cdot}{\overline{\sqrt{ax-yy}}}$. *And thence* $3x^2\dot{x} - \dot{x}yy - 2xy\dot{y} + \frac{a^3\dot{x} - 2aay\dot{y}}{2\sqrt{ax-yy}} = o$. *And by the Operation repeated you proceed to ſecond and third Fluxions, and ſo on. After the ſame manner, if the Æquation was* $x^3 - xyy + aa\overset{\cdot}{\overline{\sqrt{ax-yy}}} - b^3 = o$, *Where if you would exterminate the Fluxion* $\overset{\cdot}{\overline{\sqrt{ax-yy}}}$, *make* $\sqrt{ax-yy} = z$, *and you'll have* $ax - yy = z^2$: *And (by this Propoſition)* $a\dot{x} - 2y\dot{y}$ *will be* $= 2z\dot{z}$ *or* $\frac{a\dot{x} - 2y\dot{y}}{2z} = \dot{z}$, *that is* $\frac{a\dot{x} - 2y\dot{y}}{2\sqrt{ax-yy}} = \overset{\cdot}{\overline{\sqrt{ax-yy}}}$, *and thence* $3x^2\dot{x} - \dot{x}yy - 2x\dot{y} + \frac{a\dot{x} - 2y\dot{y}}{2\sqrt{ax-yy}} = o$. *And by the Operation repeated you may proceed to ſecond and third, and any degree of following Fluxions.*

Let the Æquation be $zy^3 - z^4 + a^4 = o$, *and the firſt Operation will produce*

Firſt Fluxions.	$\dot{z}y^3$	$+3z\dot{y}y^2$	$-4\dot{z}z^3 = o.$
The Second Operation, Second Fluxions.	$\ddot{z}y^3$	$3\dot{z}\dot{y}y^2$	$-4\ddot{z}z^3$
	$3\dot{z}\dot{y}y^2$	$6z\dot{y}^2y$	$-12\dot{z}^2z^2$
		$3z\ddot{y}y^2$	

The Sum.	$\ddot{z}y^3$	$+6\dot{z}\dot{y}y^2$	$+6z\dot{y}^2y$	$+3z\ddot{y}y^2$	$-4\ddot{z}z^2$	$-12\dot{z}^2z^2$
Third Fluxions	$\dddot{z}y^3$	$+6\ddot{z}\dot{y}y^2$	$+6\dot{z}\dot{y}^2y$	$+3\dot{z}\ddot{y}y^2$	$-4\dddot{z}z^3$	$-12\ddot{z}\dot{z}z^2$
	$3\ddot{z}\dot{y}y^2$	$+12\dot{z}\dot{y}^2y$	$+12z\ddot{y}\dot{y}y$	$+6z\ddot{y}\dot{y}y$	$-12\ddot{z}\dot{z}z^2$	$-24\dot{z}^3z$
		$+6\dot{z}\ddot{y}y^2$	$+6z\dot{y}^3$	$+3z\dddot{y}y^2$		

The Sum $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 4\dddot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}^3z = 0$.

But when you proceed from firſt to ſecond, third, and ſubſequent Fluxions, it will be convenient to conſider ſome one Quantity as flowing uniformly, and to ſubſtitute Unity in the room of its firſt Fluxion, and o *for the ſecond and following ones. Suppoſe you have the ſame Æquation as above,* viz.

$$zy^3 - z^4 + a^4 = 0,$$

and z *flows uniformly, and for its firſt Fluxion put Unity, and you'll have by the firſt* Operation $y^3 + 3z\dot{y}y^2 - 4z^3 = 0$; *by the ſecond* $3\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 12z^2 = 0$; *by the third* $9\ddot{y}y^2 + 18\dot{y}^2y + 3z\dddot{y}y^2 + 18z\dot{y}\ddot{y}y + 6z\dot{y}^3 - 24z = 0$.

But in theſe ſorts of Æquations, you muſt make (or conceive) the Fluxions to be all of them in each of the Terms of the ſame order, that is, all firſt Fluxions, as $\dot{y}$, $\dot{z}$, *&c. or all ſecond Fluxions, as* $\ddot{y}$, $\dot{y}^2$, $\dot{y}\dot{z}$, $\dot{z}^2$, *&c. or all third Fluxions, as* $\dddot{y}$, $\ddot{y}\dot{y}$, $\ddot{y}\dot{z}$, $\dot{y}^3$, $\dot{y}^2\dot{z}$, $\dot{y}\dot{z}^2$, $\dot{z}^3$, *&c. And where the Æquation is not ſo, the Order of the Fluxions is to be fill'd up by the ſuppos'd Fluxions (as if they were actually ſet down) of a Quantity flowing uniformly. Thus the laſt Æquation, by filling up the deficient Fluxions, becomes*

$$9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z\dot{z}^3 = 0.$$

And by the ſame Methods as we have made uſe of above for radical Quantities, the Fluxions of Fractions may be found; *viz.* If it was propos'd to find the Fluxion of this Fraction $\frac{a}{x}$ or of this $\frac{x}{y}$, *&c.* by putting 1°, $\frac{a}{x} = z$, you'll have for the Fluxion $\frac{a\dot{x}}{xx} = \dot{z}$, and for the other $\frac{y\dot{x} - x\dot{y}}{yy} = \dot{z}$, and ſo for others.

Now for the Fluxions of Logarithmical Quantities, the Fluxion of $l : x$ (where note l ſtands for the Logarithm of x) will be $\frac{\dot{x}}{x}$, and the Fluxion of $l\overline{x+1}$ will be $\frac{\dot{x}}{x+1}$ &c. But we ſhall have a fuller and more ample occaſion to treat of theſe in the ſeventh Chapter; whereſore we ſhall ſay no more of them here. The Fluxion alſo of R^{λ} will be $= \lambda\dot{R}R^{\lambda-1}$; where if λ be a Fraction, the Index will be negative; *i. e.* as we have before taken notice, it will change the Quantity to which it is prefix'd into a Diviſor. Thus *v. g.* To find the Fluxion of $az^{\theta} \times \overline{e+fz^{n}}|^{\lambda} = QR^{\lambda}$; you'll have $\dot{Q}RR^{\lambda-1} + \lambda Q\dot{R}R^{\lambda-1}$, that is $\overline{\dot{Q}R + \lambda Q\dot{R}}| \times R^{\lambda-1}$ which is eaſily found in explicit or common Terms; where if λ be leſs than Unity, $R^{\lambda-1}$ will of courſe become a Diviſor, *e. g.* Suppoſe it be $\overline{e+fz^{n}}|^{\frac{1}{2}}$ its Fluxion will be $nf\dot{z}z^{n-1} \times 2 \times \overline{e+fz^{n}}|^{-\frac{1}{2}} = \frac{nf\dot{z}z^{n-1}}{2\sqrt{e+fz^{n}}}$ * and ſo for the reſt.

By the direct method, from the given analytic Expreſſion of Areas of Curves, or from the Content of a curvilinear Solid, generated by the Rotation of a Plane about its own Axis, *&c.* you may obtain the Ordinates of the Curves, as alſo their eſſential Properties, the Tangents of Curves, and ſeveral other things too long to inſiſt on here.

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* *Some Examples hereof may be ſeen in the Sequel of this Treatiſe.*

CAP.

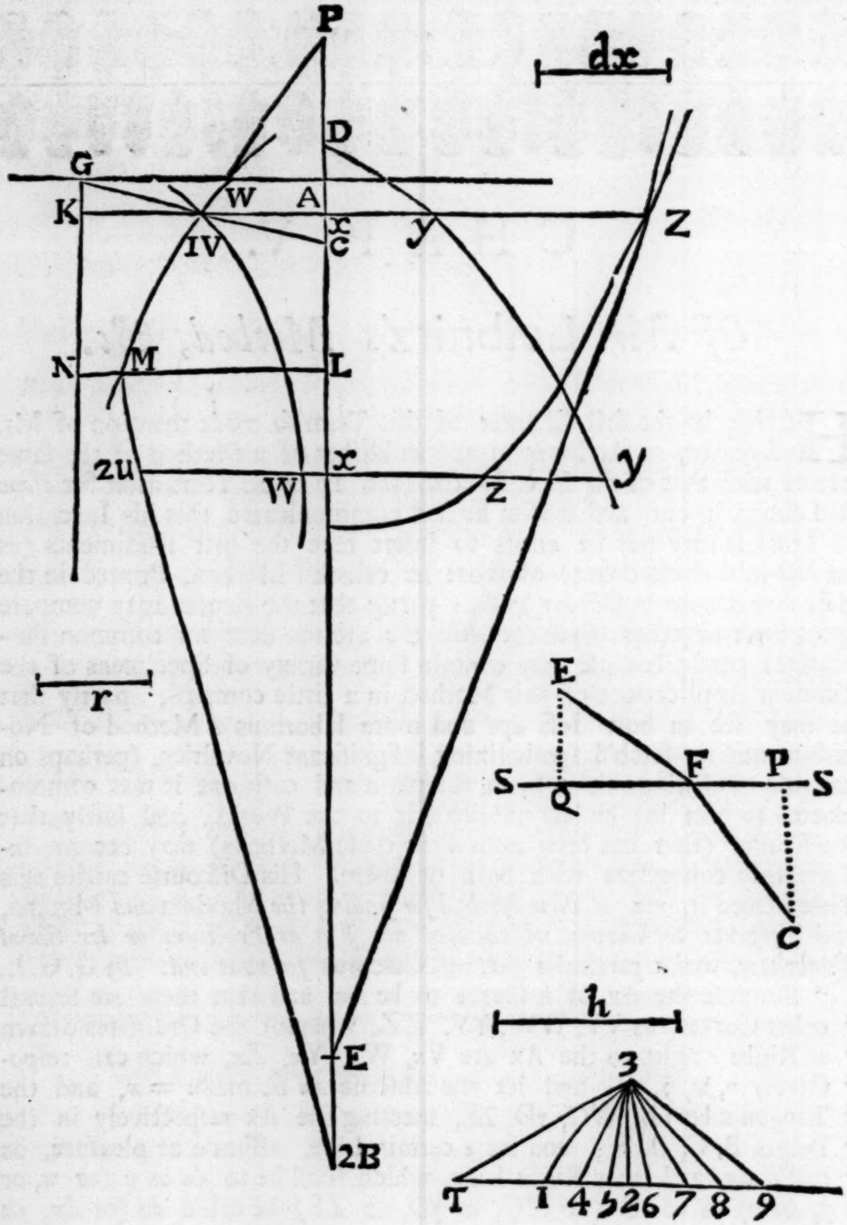
CHAP. V.

Of Mr. Leibnitz'*s Method, &c.*

HAving in the firſt Chapter of this Treatiſe made mention of Mr. *Leibnitz,* as the Inventor and Publiſher of a Method of the ſame nature with that of Sir *Iſaac Newton* (tho' 19 or 20 Years after Sir *Iſaac* had found it out, and 8 after he had communicated this his Invention to him) it may not be amiſs to inſert here the firſt Rudiments (as he has laid them down) of what he calls his Method, Printed in the *Acta Eruditorum* in *October* 1684; partly that the Reader may compare them both together (ſince the *Acta, &c.* are too dear for common Purchaſers) partly becauſe they contain ſome variety of Specimens of the different Applications of this Method in a little compaſs, partly that he may ſee in how leſs apt and more laborious a Method of Notation, and far-fetch'd ſymbolizing inſignificant Novelties, (perhaps on purpoſe to diſtinguiſh it from the plain and eaſy one it was communicated to him in) he has publiſh'd it to the World, and laſtly that the Reader (that has been unus'd to theſe Methods) may become indifferently converſant with both of them. His Diſcourſe carries this Title before it, *viz. A New Method for finding the* Maxima *and* Minima, *and Tangents to Curves, which does not ſtop at Fractions or Irrational Quantities, and a particular ſort of* Calculus *for that end. By* G. G. L.

“ Suppoſe the Ax of a Curve to be Ax, and that there are ſeveral
“ other Curves, as VV, WW, YY, ZZ, whereof the Ordinates drawn
“ at Right Angles to the Ax are Vx, Wx, Yx, Zx, which call reſpectively
“ v, w, y, z; and let the Abſciſſe ax be made $= x$, and the
“ Tangents be vB, WC, yD, ZE, meeting the Ax reſpectively in the
“ Points B, C, D, E; and let a certain Line, aſſum'd at pleaſure, be
“ call'd dx, and let a Right Line, which ſhall be to dx as v (or w, or
“ y, or z) is to vB (or WC, or yD, or ZE) be call'd dv (or dw, or

dy,


P
dx
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“ dy, or dz) or the Difference of the v’s (or of the w’s, or of the y’s,
“ or z’s) theſe things being ſuppos’d, the Rules of the *Calculus* will
“ be as follow:

“ Suppoſe a to be a conſtant given quantity, da will be $= 0$;
“ and $d,ax = adx$, if y is $= v$ (or any Ordinate of the Curve yy equal
“ to any other correſpondent Ordinate of the Curve vv (dy will be
“ $= dv$.) Now for Addition and Subſtraction, if $z - y + vv + x = v$,
“ you’ll have $d,\overline{z - y + vv + x}$, or $dv = dz - dy + dvv + dx$. For
“ Multiplication, $d : xv$ is $= xdv + vdx$, or ſuppoſing $y = xv$, you’ll
“ have $d,y = xdv + vdx$: for you may at pleaſure either make uſe of
“ the Formula xv, or ſubſtitute any Letter in the room of it, as y.
“ The Reader is here to take notice, that we manage x and dx after
“ the ſame method as we do y and dy, or any other Letter we ſub-
“ ſtitute for an indeterminate Quantity, with its reſpective *Differential*;
“ and you are to obſerve beſides, that you cannot always
“ go from a [*fluxionary or*] differential Æquation to an integral one,
“ without having occaſion for ſome caution, as we ſhall ſhew here-
“ after. Moreover for Diviſion, $d\frac{v}{y}$ (or making $z = \frac{v}{y}$) the dz is
“ $= \frac{\pm vdy \mp ydv}{yy}$. But you are to take particular care of the Signs, when
“ in the *Calculus* for any Letter you ſet down only ſimply its Differential,
“ the ſame Signs are preſerv’d, and for $+ z$ you write $+ dz$,
“ for $- z$ you write $- dz$, as is apparent from the Addition and
“ Subſtraction above: But when you proceed to the Exegeſis of the Values,
“ or when you come to conſider the relation of z to x, then it
“ will appear whether the Value of dz be an affimative Quantity, or
“ one leſs than nothing or negative, which latter when it happens,
“ then the Tangent zE is drawn from the Point z, not towards A,
“ but on the contrary ſide, or below x, that is, then when the Ordinates
“ z decreaſe, the x’s increaſing; and becauſe the Ordinates v ſometimes
“ increaſe and ſometimes decreaſe, dv will be one while an affirmative
“ and another a negative Quantity, and in the former caſe the Tangent
“ 1V1B is drawn towards A, in the latter 2V2B is drawn on the
“ contrary ſide, or the contrary way; but neither the one nor the other
“ is done in the middle about M in which moment the v’s neither increaſe
“ nor decreaſe, but remain in the ſame ſtate, and conſequently dv
“ $= 0$; where it matters not whether the Quantity be affirmative or negative;
“ For $+ 0 = - 0$; and in that place v, *viz.* the Ordinate *l*M
“ is the greateſt (or the leaſt if the Convexity of the Curve tends to-
“ wards the Ax) and the Tangent of the Curve in M is neither drawn

“ above X towards A, and there approaches the Ax, nor yet below X towards the contrary part, but is parallel to the Ax. If dv be infinite in reſpect of dx, then the Tangent falls at right Angles to the Ax, or is an Ordinate. If dv and dx are equal, the Tangent makes an Angle of 45°. with the Ax. If the Ordinates v increaſing, their Increments or Differences dv alſo increaſe (or if dv being made affirmative, alſo ddv, or the ſecond Differences, are affirmative too, or the one being negative the others are ſo too) the Curve turns its concave ſide to the Ax, otherwiſe its convex one: but where the greateſt or leaſt Increment chances to fall, or where the Increments from decreaſing ones become increaſing, or the contrary, there is the Point of the *Contrary Flexure*, and the Concavity and Convexity are chang'd, ſuppoſing the Ordinates there from increaſing ones not to become decreaſing, or the contrary, for then the Concavity or Convexity would remain; but it is impoſſible that the Increments ſhould continue to increaſe or decreaſe, and the Ordinates not from increaſing ones become decreaſing, or contrarywiſe: Therefore the Point of contrary Flexure obtains, when neither v nor dv being 0, yet ddv is 0. Whence alſo the Problem of contrary Flexure has not two equal Roots, as the Problem of the *Maxima* has, but three, and all this depends upon the right uſe of the Signs. Sometimes we are to make uſe of ambiguous Signs, as lately in Diviſion, that is, before it appears how they are to be explain'd, and if the x's increaſing, the $\frac{v}{y}$'s increaſe (or decreaſe) the ambiguous Signs in the $d\frac{v}{y}$ or in $\frac{\pm vdy \pm ydv}{yy}$, muſt be ſo explain'd that this Fraction may be an affirmative (or negative) quantity, but $\mp$ ſignifies the contrary of $\pm$, as if this be $+$ that muſt be $-$, or the contrary. There may moreover occur ſeveral Ambiguities in the ſame *Calculus*, which I diſtinguiſh by Parentheſes, for example, if there was $\frac{x}{y} + \frac{y}{z} + \frac{x}{v} = w$, you'd have $\frac{\pm vdy \mp ydv}{yy} + \frac{(\pm)\, ydz (\mp)\, zdy}{zz} + \frac{(\pm) xdv (\mp) vdx}{vv} = dw$; otherwiſe the Ambiguities ariſing from different Heads would be confounded: Where note, an ambiguous Sign multiply'd by itſelf gives $+$, and by its contrary $-$; and multiply'd by another ambiguous Sign it gives a new Ambiguity depending on the other two.

“ The Differential of the Power x^a is $= ax^{a-1}\, dx$, for example, $d\overline{x}^3 = 3x^2\, dx$; $d\frac{1}{x^a} = \frac{adx}{x^{a+1}}$; if $w = \frac{1}{x^3}$, you will have $dw = -\frac{3dx}{x^4}$; and for Roots the $d\sqrt[b]{x^a} = \frac{a}{b}\, dx \sqrt[b]{x^{a-b}}$, (hence $d\sqrt[2]{y} = \frac{dy}{2\sqrt[2]{y}}$; for in this

“ caſe a is (1) and b (2) therefore $\frac{a}{b}\sqrt[b]{x^{a-b}}$ is $\frac{1}{2}\sqrt[2]{y^{-1}}$, now y^{-1}
“ is the ſame as $\frac{1}{y}$ from the nature of the Exponents of a Geome-
“ trical Progreſſion, and the $\sqrt[2]{\frac{1}{y}}$ is $\sqrt[\frac{1}{2}]{\frac{1}{y}}$) $d\frac{1}{\sqrt[b]{x^a}} = -\frac{bdx}{a\sqrt[b]{x^{b-a}}}$;
“ But the Rule for an Integer Power might have ſufficed both for
“ Fractions and Roots, for a Power is a fractional one when its
“ Index is negative, and is changed into a Root when its Expo-
“ nent is a Fraction, but I had rather deduce theſe Conſequences my
“ ſelf, than leave them to be deduc'd by others, ſince they are general
“ and occur frequently; and in a matter difficult enough of itſelf, it
“ is not amiſs to conſult the Readers Eaſe. Now from this being
“ known as the Algorithm, as I may ſay, of this *Calculus*, which
“ I call differential, all other differential Æquations may be found by
“ the common *Calculus*, and the *Maxima* and *Minima* and *Tangents* found
“ without any occaſion of exterminating Fractions, or Surds, or any
“ Vincula, which yet was neceſſary according to the Methods * hither-
“ to publiſh'd. The Demonſtration of them all will be eaſy to a
“ Perſon converſant in theſe Methods, particularly to one that conſi-
“ ders what has not hitherto been enough thought of, *viz.* that the
“ *dx*'s, *dy*, *dv*, *dw*, *dz*'s, may be conſider'd as proportional (each in its
“ Series *or Place)* to the Differences, or momentaneous Increments or
“ Decrements of *(their reſpective Indeterminates)* x, y, v, w, z, whence
“ having any Æquation propos'd, its differential Æquation may be had,
“ by ſubſtituting ſimply its reſpective Differential for each Member (that
“ is, for each part of it that concurs to the making up the whole
“ Æquation only by Addition or Subſtraction) and by uſing for any
“ other Quantity (which it ſelf is not a Member, but concurs to the
“ making up a Member) its Differential, to form the Differential of
“ the Member; and that not ſimply, but according to the Algorithm
“ already preſcrib'd; the Methods already in uſe have not the ſame
“ Exit, for they generally make uſe of a right Line as DX, or ſome
“ other of the like nature, but not ſuch an one as *dy*, which is a fourth
“ Proportional to D*x*, *dy*, and *dz*, which confounds the whole Proceſs.
“ Hence they order, that you ſhould firſt exterminate the Fractions and
“ irrational Quantities, which are involv'd with the indeterminate ones.

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* *Here he ſpeaks as if he had been conſcious that there was ſome unpubliſh'd Method whereby the ſame was perform'd, which was that Sir* Iſaac Newton *had given him notice of.*

“ But it is evident that our Method extends itſelf alſo to *Tranſcendent*
“ *Lines, which cannot be expreſs'd by any* (purely) *Algebraic* Calculus,
“ *or which are of no certain Degree or Dimenſion*, and that univerſally,
“ without any particular Suppoſitions which do not generally ſuc-
“ ceed, only remembring this in general, That to find a Tangent is to
“ draw a right Line that ſhall join or touch two Points of a Curve, that
“ are at an infinitely ſmall Diſtance from each other, or that make
“ the Side, when produc'd, of an infinit-angular Polygon, which we
“ conſider as equivalent to a Curve; but that infinitely ſmall Difference
“ may always be expreſs'd by ſome differential Note, as dv, or by
“ ſome Quantity related to it, that is, by ſome known Tangent; eſpe-
“ cially if y was a tranſcendent Quantity, as for example the Ordinate
“ of a Cycloid, and that was to enter the *Calculus*, by the help where-
“ of the Ordinate z of another Curve was determin'd, and you
“ were to find dz, or by it the Tangent of this latter Curve; as if you
“ were to determine dz by dy, and you had dy becauſe you had the
“ Tangent of the Cycloid; but if you had not yet the Tangent of the
“ Cycloid, it might in like manner be found by Calculation from the
“ given Property of the Tangents of a Circle. But I ſhall here bring
“ an Example of the *Calculus*, wherein let it be minded that I denote
“ Diviſion thus, $x : y$ which ſignifies the ſame thing as x divided by y,
“ or $\frac{x}{y}$; Suppoſe the firſt or given Æquation to be $x : \overline{y+a} + b \times \overline{c-xx}$
“ ſqu. $\frac{y}{ex+fxx} + ax\sqrt{\overline{gg+yy}} + yy : \sqrt{\overline{bb+lx+mxx}} = 0$. Expreſ-
“ ſing the relation between x and y, or between Ax and xy, ſuppoſing
“ $a, b, c, d, e, f, g, b, l, m$, to be given Quantities, to find the me-
“ thod of drawing YD from the given Point y ſo as to touch the Curve,
“ or to find the reaſon of the right Line DX to the given right Line XY;
“ for brevity ſake write n for $a + bx$, and p for $c - xx$, and q for ex
“ $+ fxx$, and r for $gg + yy$, and s for $bb + lx + mxx$; you'll have $x : y$
“ $+ np : qq + ax\sqrt{\overline{r+yy}} : \sqrt{s} = 0$; which ſuppoſe to be the ſecond Æqua-
“ tion. Now by our *Calculus* the $d, x : y$ is $+ xdy + ydx : yy$, and in like man-
“ the $d, np : qq$ is $(\pm)\, 2npdq\, (\mp)\, q\overline{dnp + pdn} : q^3$ & $dax\sqrt{r}$ is $- axdr : 2r$
“ $\overline{\sqrt{\ } + adx}\sqrt{r}$; and $dyy : \sqrt{as}$ is $((\pm))\, yyds\, ((\mp)) - 4ysdy : 2s\sqrt{s}$; all
“ which differential Quantities from $d, x : y$ to $d, yy : \sqrt{s}$, added up in-
“ to one Sum make 0, and will thus give a third Æquation, for
“ ſo you ſubſtitute their differential Quantities for the Members of the
“ ſecond Æquation. Now dn is bdx and dp is $- 2xdx$, and dq is edx
“ $+ 2fxdx$, and dr is $2ydy$, and ds is $ldx + 2mxdx$; which Values be-
“ ing ſubſtituted in the third Æquation, you have a fourth Æquation,

“ wherein the Differential Quantities which alone remain, *viz.* dx, dy,
“ are always found without Denominators or Vincula, and each Member
“ is either affected by dx or dy, the Law of Homogeneals being
“ always obſerv'd as to theſe two Quantities, how complicated ſo-
“ ever the Calculus be; whence you may always have the Value of
“ $dx : dy$, or of the reaſon of dx to dy, that is of DX ſought, to XY
“ given, which reaſon in this our Calculus, (by changing the fourth
“ Equation into an Analogy) will be as $\pm x : y + axy : \sqrt{r}\ ((\pm))\ 2y :$
“ $\sqrt{s}$ is to $\pm 1 : y\ (\pm)\ 2vpe + 2fx : q\ (\pm) - 2nx + pb : qq + a\sqrt{r}\ ((\pm))$
“ $yyl + 2mx : 2s\sqrt{s}$. But x and y are given from the given Point y,
“ and the aforemention'd Values of the Letters n, p, q, r, s are had by
“ x and y; and this Example, which is complicated enough, we have
“ only made uſe of here, that we might ſhew the Method of the above
“ deliver'd Rules in more implicit caſes than this; now we'll take
“ an opportunity of ſhewing it in caſes more obvious to the Underſtanding.
“ Let there be given the two Points C and E, and the
“ Right-Line SS in the ſame Plane with them, to find the Point F in
“ the Right-Line SS that is ſo to be taken in it, that having drawn
“ CF and FE, the Aggregate of the Rectangle CF into the given nb,
“ and of FE into the given r ſhall be the leaſt of all poſſible, that is,
“ if the Line SS ſeparate the Mediums, and b repreſents the Denſity
“ of the Medium as of Water on the part of C, and r the Denſity of
“ the Medium as of Air on the part of E, to find the Point F, ſo that
“ the way from C to E thro' F ſhall be the eaſieſt (or ſhorteſt) of
“ all ways poſſible, let us ſuppoſe all thoſe poſſible Aggregates of the
“ Rectangles, or all the Difficulties of all poſſible Ways to be repre-
“ ſented by the Ordinates KV, of the Curves VV, perpendicular to
“ the Right-Line GK, which we will call ω, and that NM the leaſt
“ of them is ſought; becauſe the Points C and E are given, there will
“ alſo be given the Perpendiculars to SS, *viz.* CP (which we will call
“ c) and FQ (which we will call e) and moreover PQ (which we will
“ call p) and QF which is equal to GN (or AX) we will call x,
“ and CF, f; & EF, g; you'll have FP, $p - x$; $f = \sqrt{cc + pp - 2px + xx}$,
“ or by a Compendium $\sqrt{l}$, and $g = \sqrt{ee + xx}$, or by a Compendi-
“ um $\sqrt{m}$, we have therefore $\omega = b\sqrt{l} + r\sqrt{m}$, the Differential of
“ which Æquation, making $d\omega = 0$ (in caſe of the Minima) is
“ $0 = -bdl : \sqrt[2]{l - rdm} : 2\sqrt{m}$ by the Rules of our Calculus; now
“ $dl - 2dx - \overline{p + x}$, and dm is $2xdx$, therefore $\overline{bp - x} : f$ is $= rx : g$;
“ Now if this be accommodaied to *Dioptricks*, and you make f, or g,
“ or CF and EF equal, becauſe there Remains the ſame Refraction in

“ the Point F, how great ſoever you make the Length of the Right-
“ Line CF, $b\overline{p-x}$ will be $= rx$, or $b : r :: x : p-x$, or b to r as QF
“ to FP; that is, the Sines of the Angles of Incidence and Refraction
“ FP and QF will be reciprocally as r and b the Denſities of the Me-
“ dia whereon the Incidence and wherein the Refraction is made.
“ Which Denſity is yet not to be eſtimated in reſpect to us, but in
“ reſpect to the Reſiſtance the Rays of Light meet with. And thus
“ you have a Demonſtration of the Calculus we have elſewhere ſhewn
“ in theſe *Acta*, where we laid down a general Foundation of *Optics*,
“ *Catoptrics*, and *Dioptrics*; while ſeveral other very learned Men
“ have ſought by ſeveral *Ambages* or round-about ways, what any
“ one ſkill'd in this Calculus may do in three Lines; which I'll ſhew
“ by another Example. Let the Curve 133 be of ſuch a nature, that
“ the ſix Right-Lines 34, 35, 36, 37, 38, 39, drawn to the ſix fix'd
“ Points in the Ax, 4, 5, 6, 7, 8, 9, from any Point of it, as 3, being
“ added together, ſhall be equal to a given Line g. Let the Ax be
“ T14526789, and 12 the Abſciſſe, 23 an Ordinate, To find the Tan-
“ gent 3T; I ſay T2 will be to 23 as $\frac{23}{34}+\frac{23}{35}+\frac{23}{36}+\frac{23}{37}+\frac{23}{38}+\frac{23}{39}$
“ is to $-\frac{24}{34}-\frac{25}{35}+\frac{26}{36}+\frac{27}{37}+\frac{28}{38}+\frac{29}{39}$. And the Rule will be
“ the ſame, only continuing the Terms on, if not [only] ſix, but ten
“ or more fix'd Points were ſuppos'd, which to manage by any of the
“ common Methods or Ways of Calculation, by exterminating the
“ irrational Quantities, according to the vulgarly known Methods of
“ Tangents, would be very tedious and ſometimes ſufficiently labo-
“ rious, as if all the Rectangles or Solids compos'd of thoſe Right-
“ Lines, by their poſſible Binaries or Ternaries were to be made equal
“ to a given Quantity; in all which caſes, and ſome a great deal harder
“ than thoſe, there is the ſame Facility or Eaſe to thoſe that work
“ according to our Calculus, even beyond Expectation. Thus we have
“ laid the Foundations of a more ſublime Geometry [than uſual] that
“ will reach to the moſt difficult and illuſtrious Problems even of the
“ Mixt Mathematics themſelves, which without our *Differential Calculus*,
“ or *ſome other like it*, none could ſo eaſily arrive to. By way of Ap-
“ pendix we ſhall here add the Solution of a Problem, which Monſ.
“ *Des Cartes* had heretofore propos'd to him by Monſ. *De Beaune*, and
“ endeavour'd at it, as may be ſeen in his third Tome of *Epiſtles*, but
“ did not ſolve, *viz.* To find a Line WW of ſuch a nature, that
“ the Tangent WC being drawn to the Ax XC ſhou'd always be
“ equal to the conſtant (or determinate) Line a. Now XW or w
“ is to XC or a, as dw to dx: therefore if dx (which may be taken
“ at pleaſure) be aſſum'd conſtant, or always the ſame, *viz.* b; or
if

" if the x's or AX's increaſe uniformly, W will be $= \frac{a}{b} d$W, which
" will be the very Ordinates W proportional to the dw's or their Increments or Differences; that is, if the x's are in an Arithmetical
" Progreſſion, the w's will be in a Geometrical one; or if the w's
" are Numbers, the x's will be Logarithms: Wherefore the Line WW
" is the Logarithmic Line.

CHAP. VI.

Of the Method of finding Fluents, &c. and particularly of the Fluents of Rational Fluxions.

SIR *ISAAC NEWTON*, in the laſt Page of his Treatiſe of *Quadratures*, gives us this Rule in general, for the finding not only the Fluents of *Rational Fluxions*, but univerſally of any whatſoever, *viz.*

A Fluent may be aſſum'd at pleaſure, and the Aſſumption corrected by putting the Fluxion of the aſſum'd Fluent equal to the Fluxion propos'd, and comparing the Homologous Terms together. Whether or not he found out thoſe excellent Theorems in the preceding Pages by this Method, (as I am apt to believe he did) he does not there inform us; but he plainly enough hints that they may be found out this way; and and brings this Rule as a Proof of the Truth of the Concluſion, ſo inferr'd, in the precedent Page. *After you have found the Fluents of the given Fluxions, if you doubt of the truth of your Concluſion, you may collect the Fluxions of the Fluents found, and compare them with the Fluxions at* *firſt given; for if they come out equal, the Concluſion is right: if not, you muſt correct your Fluents ſo as to make their Fluxions come out equal to the Fluxions originally propos'd.* And becauſe *any Fluent* (as he there remarks) *which is collected or found from a firſt Fluxion may be augmented or diminiſh'd by any* conſtant *Quantity, or ſuch as does not flow* (which in bringing out the Fluxion by the direct Method may have vaniſh'd) *as alſo any that is collected from a ſecond Fluxion, may be augmented or diminiſh'd by a Quantity that has no third Fluxion, and ſo on* ad infinitum. Now if any one has a Mind to reſtore that Quantity, Mr. *Craig* and Dr. *Cheyne* have ſhewn a Method of doing it; the Firſt in pag. 38 of his *Quadratures*, Printed in 1694, and the Other in pag. 65 of his *Inverſe Method*, and the 9th of his *Rudiments*, viz. *That Quantity may be found by making the Abſciſſe, or the Fluent itſelf as found* (by the Methods hereafter to be ſhewn) = 0, *and you'll have the Sign, and again by putting the indeterminate Quantity by which that Fluent is expreſs'd* = 0, *you will have the Quantity itſelf, that is to be added to the Fluent before found, under a contrary Sign, as the Signs direct.* You ſee the reaſon of this pag. 9 of Dr. *Cheyne*'s *Rudiments*, and an Example in pag. 65 of his *Inverſe Method.*

The Aſſumptions, or Fluents to be aſſum'd, for any ſort of Fluxions whatſoever, will eaſily enough appear from the conſideration of the Nature of the Fluxion, as Dr. *Cheyne* has hinted; and in ſimple Fluxions, or ſuch as are not involv'd under a *Vinculum*, the *Fluent* is found (to uſe the Doctor's Expreſſion) *by Operations juſt retrograde or contrary to thoſe whereby the Fluxions were brought out from their Fluents.* Thus, *e. g.* the F: $mx^{m-1}\dot{x}$ is $= \frac{m x^{m-1+1}\dot{x}}{\overline{m+1-1}\,\dot{x}} = x^m \pm a^m$, &c. And the Fluent of $my^{\pm n}x^{m-1}\dot{x} \pm nx^m y^{\pm n-1}\dot{y} = x^m y^{\pm n} \pm a^{m\pm n}$, *&c.* that is, for $mx^{m-1}\dot{x}$, aſſume Ax^m, and having found its Fluxion, put it equal to the Fluxion given, *viz.* to $mAx^{m-1}\dot{x} = mx^{m-1}\dot{x}$. Whence comparing the Terms you'll have $A = \frac{m}{m} = 1.$ and the Fluent will be $x^m \pm a^{m\pm n}$ *&c.* After the ſame way for the other, you muſt aſſume $Ax^m y^{\pm n}$ &c. Thus if there were given $\overline{m-1} \times m\dot{x}^2 x^{m-2} + m\ddot{x}x^{m-1}$, you might aſſume for its immediate Fluent $A\dot{x}x^{m-1}$ whence by the direct Method you would have $\overline{m-1}\,A\dot{x}^2 x^{m-2} + A\ddot{x}x^{m-1}$, and you would find

$\overline{m-1}A + 1A = \overline{m-1 \times m} + 1m$ & $A = \frac{\overline{m-1\times m}+1m}{m-1+1} = m$, and $A\dot{x}x^{m-1} = m\dot{x}x^{m-1}$, and ſo on *ad infinitum.* And it is always requiſite that the Fluents ſhould (if poſſible) be expreſs'd in the ſame indeterminate Quantities as the Fluxions are.

In Fluxions involv'd under *Vincula*, the Aſſumptions muſt be either general or particular, as the *Indices* direct; and as in common *Algebraic* Æquations you may aſſume a general or infinite Æquation, and (leaving out the Signs of the particular Members, to be ſupply'd by + or —, as occaſion ſhall require) ſolve that generally, either abſolutely, or by an infinite Series, in order thence to derive a Canon, by which any particular Æquation whatever may be afterwards ſolv'd; ſo you may here aſſume a general Fluent, for a general Fluxion, which ſhall exhibit the particular Fluents of all particular Fluxions, that can be contain'd under that Form (æquating the Quantities to be reſpectively æquated to one another) by a due application or compariſon of the particular Members of that particular Fluxion, with thoſe of the ſame Dimenſion in the general one, which will often ſave a great deal of Labour and Time in ſearching for particular Fluents by particular Methods. Thus Sir *Iſaac Newton*, in his Excellent *Treatiſe of Fluxions*, makes firſt ſuch general Forms for the Fluents of general Rational Fluxions, and then ſolves particular Fluxions by them, wherein Dr. *Cheyne* alſo follows him; and how much eaſier and more ready this way frequently is, than that of finding particular Fluents, any one converſant in theſe Methods may eaſily judge: For ſuch a primary and general Aſſumption is always regular, and therefore the moſt facil, and if it yields more Degrees of Infinity than one, the following Series or Aſſumptions will naturally emerge from the Fluxion of the firſt; and (as we have ſaid) by this general Fluent any particular one may be determin'd, as we ſhall ſhew hereafter.

But to return to Sir *Iſaac Newton*, and the Quadratures of Curves. It muſt be noted that there is often a very great difference between the Algebraic and genuine Fluent of a general Fluxion, and the Quadrature of a Curve, becauſe the latter may be (and often is) more limited and reſtrained than the former, as will eaſily be perceiv'd. Sir *Iſaac* in the aforeſaid Treatiſe (as Dr. *Wallis* expreſſes it) *in Curves that have parallel Ordinates, makes the Fluxion of the Abſciſſe to be uniform, and ſubſtitutes Unity for it, and puts the Ordinate for the Fluxion of the Area*; that is, putting *x* or *z* for the Abſciſſe, and having given a proper Expreſſion of the Area, *e. g.* $= v$, he finds (by the direct Method) its Fluxion $= \dot{v}$. and then making x or $z = 1$, you have the Ordinate of the Curve; and this is the deſign and ſcope of the third and fourth Propoſitions of the aforeſaid *Treatiſe of Quadratures*, after he has before ſhewn, in the ſecond Propoſition, a Method of finding ſuch Curves, as, having their Ordinates parallel, may be ſquared.

Suppoſe *e.g.* in Sir *Iſaac*'s third Propoſition R to be $= e + fz^n + gz^{2n}$ &c. and let $z^\theta R^\lambda = v =$ to the Area of a given Curve; that is $z^\theta \times \overline{e + fz^n + gz^n\ \&c.}|^\lambda = v$. Take the Fluxions on both ſides, and you'll have $\theta\dot{z}z^{\theta-1}R^\lambda + \lambda z^\theta \dot{R}R^{\lambda-1} = \dot{v}$; that is, making (after Sir *Iſaac*'s way before-mentioned) $RR^{\lambda-1} = R^\lambda$ in the firſt Member, and $zz^{\theta-1} = z^\theta$ in the ſecond, you will have $\overline{\theta\dot{z}R + \lambda z\dot{R}}|\times z^{\theta-1}R^{\lambda-1} = \dot{v}$; which in explicit Terms, putting $\dot{z} = 1$, gives $\theta e + \overline{\theta + \lambda n}\, fz^n + \overline{\theta + 2\lambda n}\, gz^{2n} + \overline{\theta + 3\lambda n}\, hz^{3n} + \&c. \times z^{\theta-1}R^{\lambda-1} =$ to the Ordinate. *Q. E. O.*

Thus in the ſecond Caſe, (or the fourth Propoſition) ſuppoſe moreover $S = k + lz^n + mz^{2n}$ &c. let $z^\theta R^\lambda S^\mu$ be $= v$. And taking the Fluxions, and ordering the Æquation as above, you will have $\overline{\theta\dot{z}RS + \lambda z\dot{R}S + \mu zR\dot{S}}|\times z^{\theta-1}R^{\lambda-1}S^{\mu-1} = \dot{v}$, which being order'd and expreſs'd in explicit Terms, making every where $\dot{z} = 1$, gives the Ordinate ſought. The ſame Method muſt be taken in further Proceſs's of this kind, as if there was alſo $T = p + qz^n + z^{2n}$ &c. and $z^\theta R^\lambda \times S^\mu T^\nu$ were $= v =$ to the Area of a Curve; and ſo on *ad infinitum*: that is, *having the Area of any Curve expreſs'd by never ſo many Series* (after this way) *to find by the Direct Method the Fluxion of the Area, (and then making the Fluxion of the Abſciſſe equal to Unity) is the ſame thing as to find the Ordinate*; whence reciprocally, *Having given the Ordinate of any Curve (that has parallel Ordinates) to multiply that into the Fluxion of the Abſciſſe, and then to find the Fluent, is the ſame thing as to ſquare the Curve.* Now having ſuch a Fluxion given, you muſt aſſume (according to Sir *Iſaac Newton*'s Rule) a Fluent, of ſuch a nature as *its Fluxion may compare with the Fluxion given*; but here you are to note, that the compariſons of this nature are made between Terms that are homologous, that is, of the ſame Algebraic Dimenſion; I ſay, you muſt aſſume ſuch a Fluent affected with indeterminate Coefficients, whoſe Fluxion, being found, may compare with the Terms of the theFluxion given, that thereby the aſſum'd Coefficients may be determined, which determined Coefficients, being ſubſtituted in the room of the indeterminate ones in the aſſum'd Æquation, will give the Fluent ſought. Of the ſame tendency are the 5 and 6 Propoſitions. For example, ſuppoſe in Prop. 5. the Ord. $= z^{\theta-1} \times R^{\lambda-1} \times \overline{a + bz^{n} + cz^{2n}}$ &c. (making as before $R = e + fz^{n} + gz^{2n}$ &c.) or which is the ſame thing, let it be expreſs'd thus, $R^{\lambda-1} \times \overline{az^{\theta-1} + bz^{\theta+n-1} + cz^{\theta+2n-1}}$ &c.

Then will $R^{\lambda-1} \times \overline{a\dot{z}z^{\theta-1} + b\dot{z}z^{\theta+n-1} + c\dot{z}z^{\theta+2n-1} + d\dot{z}z^{\theta+3n-1}}$ &c. = to the Fluxion of the Area, or (as Dr. *Cheyne* in this caſe alſo calls it) to the *Fiuxion given.* Now make the Series $a\dot{z}z^{\theta-1} + b\dot{z}z^{\theta+n-1}$ &c. $= \dot{q}$, then will $\dot{q}R^{\lambda-1}$ be = to the Fluxion given. Aſſume $Az^{\theta} + Bz^{\theta+n} + Cz^{\theta+2n}$ &c. $= Q =$ to the Rational part of the Fluent affected with indetermined Coefficients; which being multiply'd into $\overline{e + fz^{n} + gz^{2n}}|^{\lambda}$ gives the Fluent to be aſſumed = to QR^{λ}. Take (by the Direct Method) the Fluxion of this aſſum'd Fluent, *viz.* $\dot{Q}R^{\lambda}$ $\lambda Q\dot{R}R^{\lambda-1} = \dot{q}R^{\lambda-1}$, that is, when reduc'd to a Form proper for Compariſon $\overline{\dot{Q}R + \lambda Q\dot{R}}|R^{\lambda-1} = \dot{q}R^{\lambda-1}$ which being divided by the irrational part on both ſides, leaves this Canon, *viz.* $\dot{Q}R + \lambda Q\dot{R} = \dot{q}$; which being brought into explicit Terms, and compared, determines the aſſum'd Coefficienrs; which ſubſtituted in the room of the indetermined ones, will give the Fluent ſought: *e. g.* in Sir *Iſaac*'s fifth Propoſitition.

$\overline{\dot{Q}R + \lambda Q\dot{R}}| \times R^{\lambda-1} = \dot{z}z^{\theta-1} \times \overline{a + bz^{n} + cz^{2n}}$ &c. $\times \overline{e + fz^{n} \text{ \&c.}}|^{\lambda-1}$; and dividing both ſides by $R^{\lambda-1} = \overline{e + fz^{n} \text{ \&c.}}|^{\lambda-1}$, you will have $\dot{Q}R + \lambda Q\dot{R} = \dot{q} = a\dot{z}z^{\theta-1} + b\dot{z}z^{\theta+n-1} + c\dot{z}z^{\theta+2n-1}$ &c. that is, in explicit Terms,

$$\theta eA\dot{z}z^{\theta-1} + \left.\begin{matrix} \overline{\theta + n}eB \\ \theta Af \\ \lambda nAf \end{matrix}\right\} \dot{z}z^{\theta+n-1} \text{ \&c.} = a\dot{z}z^{\theta-1} + b\dot{z}z^{\theta+n-1} \text{ \&c.}$$

But hence 1. $\theta eA = a$; whence $A = \frac{a}{\theta e}$.

2°. $\theta + n$, $eB + \theta Af + \lambda nAf = b$ ſeu $\overline{\theta + n}$, $eB = b - \theta Af - \lambda nAf$.

Whence $B=\frac{b-\overline{\theta-\lambda n},fA}{\overline{\theta+n},e}$, make now the like Subſtitutions as Sir *Iſaac* does, and place them in their proper places, and you'll eaſily obtain the Area ſought expreſs'd in Sir *Iſaac*'s Terms. *Q. E. I.*

Of the ſame nature is the *Infinitinomial* Caſe, *pag.* 12. of Dr. *Cheyne* (as being the ſame with this of Sir *Iſaac) viz.* making $z^{r-1}\dot{z}\times\overline{a+bz^{n}+cz^{2n}\ \&c.}=\dot{q}$, & $\overline{e+fz^{n}\ \&c.}|^{m-1}=R^{m-1}$. Aſſume as above (making firſt $\frac{1}{p}=m$) $QR^{\frac{1}{p}}$; then will $p\dot{Q}R+Q\dot{R}=\dot{q}=paz^{r-1}+pbz^{r+n-1}$ &c. *ad infinitum.* putting $\dot{z}=1$. And the *Binomial* Fluxion of $\overline{e+fz^{n}}|^{\lambda+1}=R^{\frac{1}{p}}$ (making $\frac{1}{p}=\lambda+1$) gives us for compariſon $p\dot{Q}R+Q\dot{R}=pdz^{\theta}$; And you are to proceed after the ſame way for *Trinomial* and *Quadrinomial, &c.* Fluxions, *ad infinitum* (or to an *Infinitinomial*, the higheſt in this kind poſſible) *viz.* by making all the Coefficients after the laſt of the particular given *Nome* = 0 in the *Infinitinomial* Theorem; for every inferior Form is reſpectively contain'd in any ſuperior one of its kind.

Suppoſe now, as in the ſixth Propoſition of Sir *Iſaac*, the given Fluxion to be $\dot{q}R^{\lambda-1}S^{\mu-1}$, or $\overline{\dot{Q}RS+\lambda Q\dot{R}S+\mu QR\dot{S}}\times R^{\lambda-1}S^{\mu-1}$ or $\dot{Q}RS+\lambda Q\dot{R}S+\mu QR\dot{S}=\dot{q}$ for compariſon; having, after the ſame manner explicated and compar'd the Terms, and ſubſtituted as requiſite, you'll have the Fluent ſought. After the ſame manner, if you were to proceed *ad infinitum*, and made $T=p+qz^{n}+rz^{2n}+sz^{3n}$ &c. and the given Fluxion was

$\dot{q}R^{\lambda-1}$

$qR^{\lambda-1}S^{\mu-1}T^{\nu-1}$ you muſt in like manner aſſume $QR^{\lambda}S^{\mu}T^{\nu}$ for the Fluent, and ſo on *ad infinitum.* And hence may be ſeen what Sir *Iſaac* heretofore hinted to Dr. *Wallis*, viz. *That he could find nothing general in quadratures, till he had reduc'd the Buſineſs to the ſole Conſideration of Ordinates.* He has given us a general Demonſtration of theſe Quadratures by putting the Sum of the particular Ordinates (found by his Method) equal to the general Ordinate of a Curve conſider'd as made up of all of them, and by equalling the correſpondent Terms (*i. e.* by comparing the homologous Terms) in order to determine the aſſum'd Coefficients, and having made the proper Subſtitutions, there will come out the Area ſought.

As for Fluxions of inferior Orders, an Infinity of which are contain'd under each of theſe general Forms, whether *Bivincular* or *Trivincular, &c. ad infinitum*; Or whether *Binomial* or *Trinomial* ones under thoſe again, *&c. ad inf.* You muſt aſſume an Æquation, or rational Series adfected by indeterminate Coefficients, and multiply'd into each of thoſe Names augmented by Unity, and proceed as above for the Fluent. Examples hereof may be ſeen in Dr. *Cheyne*'s Book, pag. 9. for the *Newtonian* Binomial, pag. 68. for the Trinomial, *&c.* where alſo are Rules for the Aſſumption of the Indices. But to return to Sir *Iſaac*, he gives us theſe Rules for the Praxis.

† *Every Ordinate* (ſays he, pag. 179.) *may be reſolv'd too Ways into a Series.*

1. *Where the Index* n *is affirmative.*
2. *Where it is Negative.*

As ſuppoſe you have the Ordinate $\frac{3k - lzz}{zz\sqrt{kz - lz^3 + mz^4}}$ *this may be writ thus,*

viz. $z^{-\frac{5}{2}} \times \overline{3k - lzz} \times \overline{k - lz^2 + mz^3}|^{-\frac{1}{2}}$, *or thus* $z^{-2} \times \overline{-l + 3kz^{-2}} \times \overline{m - lz^{-1} + kz^{-3}}|^{-\frac{1}{2}}$.

Which a Learner may thus eaſily apprehend; firſt, if he makes $\frac{3k - lz^2}{z^2\sqrt{kz - lz^3 + mz^4}} = y$, then will $3k - lz^2 = yz^2\sqrt{kz - lz^3 + mz^4}$, &

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† *See alſo* Cheyn. *p.* 66.

$3k - lz^2 = yz^{\frac{4}{2}} \times z^{\frac{1}{2}}\sqrt{\overline{k - lz^2 + mz^3}} = yz^{\frac{5}{2}} \times \sqrt{\overline{k - lz^2 + mz^3}}$ that is $z^{-\frac{5}{2}} \times \overline{3k - lz^2} \times \overline{k - lz^2 + mz^3}\rvert^{\frac{1}{2}} = y$. which is the firſt. Or alſo thus $\frac{3k - lzz}{zz\sqrt{\overline{kz - lz^3 + mz^4}}} = z^2\sqrt{z}\sqrt{\overline{k - lzz + mz^3}} = \sqrt{z^4}\sqrt{z}\sqrt{\overline{k - lzz + mz^3}} = \frac{3k - lz^2}{\sqrt{z^5}\sqrt{\overline{k - lz^2 + mz^3}}}$, that is $\frac{3k - lzz}{\sqrt{z^5}\sqrt{\overline{k - lz^2 + mz^3}}} = y = z^{-\frac{5}{2}} \times \overline{3k - kz} \times \overline{k - lzz + mz^3}\rvert^{-\frac{1}{2}}$. Secondly, $z^{-\frac{5}{2}} \times \overline{3k - lz^2} \times \overline{k - lz^2 + mz^3}\rvert^{-\frac{1}{2}} = z \times \overline{3kz^{-2} - l} \times \overline{kz^{-3} - lz^{-1} + m}\rvert^{-\frac{1}{2}}$. Which will appear thus

1°. $z^{-\frac{5}{2}} \times \overline{3k - lz^2} = \frac{3k - lzz}{\sqrt{z^5}} = \frac{3k - lzz}{zz\sqrt{z}} = \frac{3kz^{-2} - l}{\sqrt{z}}$.

2°. $\overline{kz - lz^2 + mz^3}\rvert^{-\frac{1}{2}} = \sqrt{z^3} \times \overline{kz^{-3}\ lz^{-1} + m}\rvert^{-\frac{1}{2}} = \frac{\sqrt{z^3}}{\overline{kz^{-3} - lz^{-1} + m}\rvert^{\frac{1}{2}}}$.

3°. $\frac{3kz^{-2} - l}{\sqrt{z}} \times \frac{\sqrt{z^3}}{\overline{kz^{-3} - lz^{-1} + m}\rvert^{\frac{1}{2}}} = \frac{\sqrt{z^3} \times \overline{3kz^{-2} - l}}{\sqrt{z} \times \overline{kz^{-3} - lz^{-1} + m}\rvert^{\frac{1}{2}}} =$

4°. $\frac{\sqrt{z^3}}{\sqrt{z}} \times \frac{\overline{3kz^{-2} - l}}{\overline{kz^{-3} - lz^{-1} + m}\rvert^{\frac{1}{2}}} = \frac{\sqrt{z^2}}{1} \times \frac{3kz^{-2} - l}{\overline{kz^{-3} - lz^{-1} + m}\rvert^{\frac{1}{2}}} =$

5°. $z^{-2} \times \overline{3kz^{-2} - l} \times \overline{kz^{-3} - lz^{-1} + m}\rvert^{-\frac{1}{2}}$. Which is the ſecond.

And by theſe or the like Ways, the Ordinates will be reduced to a Form proper for the Cannon; *And then* (ſays Sir *Iſaac*) *you muſt try each Caſe if either of the Series breaks off, or is terminated, for want of Terms going on further, you'll have the Area in finite Terms.* Suppoſe now the aforeſaid Ordinate reduc'd to be $z^{-\frac{5}{2}} \times \overline{3k - lz^2} \times \overline{k - lz^2 + mz^3}\rvert^{\frac{1}{2}}$ which make equal to the general Ordinate in the Canon, *viz.* $z^{-\frac{5}{2}} \times \overline{3k - lz^2} \times \overline{k - lz^2 + mz^3}\rvert^{\frac{1}{2}} = z^{\theta - 1} \times \overline{a + bz^n + cz^{2n}}$ &c.

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$\times \overline{e + fz^{n} + gz^{2n} + hz^{3n} \&c.}|^{\lambda-1}$; Whence you'll have

$$\left.\begin{array}{l} \theta - 1 = -\tfrac{5}{2}.\ \theta = -\tfrac{3}{2} = r. \\ \lambda = \tfrac{1}{2}.\ \lambda - 1 = -\tfrac{1}{2}.\ n = 1. \\ s = -1.\quad t = -\tfrac{1}{2}.\ v = 0. \end{array}\right\} \begin{array}{ll} 3k = a. & k = e. \\ 0 = b. & 0 = f. \\ -l = g. & h = m \\ & \&c. \end{array}$$

and the Area will be $z^{\theta}\, R^{\lambda} \times \frac{\frac{1}{n}\, a}{re}$. Whence

$$R = k - lz^{2} + mz^{3} = e + (\,0 = f\,)\, z^{n} + gz^{2n} + hz^{3n}.$$

$$\& \; z^{\theta}\, R^{\lambda} = z^{-\frac{3}{2}} \times \overline{k - lz^{2} + mz^{3}}|^{\frac{1}{2}}.$$

$$\& \; \frac{\frac{1}{n}\, a}{re} = \frac{\frac{1}{1}\, 3k}{-\frac{3}{2}k} = -\frac{3}{2}\Big)\frac{3}{1}\Big(-\frac{6}{3} = -2.$$

And the Area will be $z^{\theta}\, R^{\lambda} \times \frac{\frac{1}{n}\, a}{re} = -2\, \frac{\sqrt{k - lz^{2} + mz^{3}}}{z^{3}}$. *Q. E. I.*

* *And this Area expreſſes the lower part of the Curve. For every Affirmative Area expreſſes the upper part of the Curve, and the Negative one falls on the contrary ſide of the Ordinate, and goes on with the Axis produc'd, the Sign of the Ordinate remaining as before. And thus always one of the Series, and ſometimes both, is terminated, and becomes finite, if the Curve can be geometrically ſquared, but if it does not admit of ſuch a Quadrature, each of the Series will continue on* ad inf. *and one of them will converge, and give the Area by Approximation, except,* 1. Where $\frac{\theta}{n} = r$ (by reaſon of an infinite Area) is either nothing, or an Integer and negative Number: Or 2. When $\frac{z^{n}}{e} = 1$. *i. e.* when $z^{n} = e$; which are the Conditions of Non convergency; the Conditions of converging are

1. If $\frac{z^{n}}{e}$ is leſs than Unity; and then that Series will converge wherein the Index n is Affirmative; and in this Caſe the Area is on the upper Part of the Curve.

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* *Pag* Neut. *Edit. Anglic.* 179. *Pag.* Cheyn. 66.

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2. But if $\frac{z^{n}}{e}$ be greater than Unity, the other Series will converge; and the Area lies on the lower part of the Curve.

There follow in Sir *Iſaac*, ſome Notes relating to Ordinates, ſuch as theſe.

Suppoſe in any Ordinate P = to the Numerator.

$Q = \left\{\right.$ To any rational Factor or Denominator.

R^{π} = To an irreducible ſurd Factor.

R = To its Side.

S^{μ} and S as alſo T^{ν} and T = to the ſurd Factors, and their Roots.

Hence the Forms of the Ordinates.

The First proceeding on ad inf.	*The Second proceeding on ad inf.*
1. QR^{π}	1. $\frac{P}{Q}$
2. $QR^{\pi}S^{\mu}$	2. $\frac{P}{QR^{\pi}}$
3. $QR^{\pi}S^{\mu}T^{\nu}$ *&c. ad inf.*	3. $\frac{P}{QR^{\pi}S^{\mu}}$
	4. $\frac{P}{QR^{\pi}S^{\mu}T^{\nu}}$ *&c. ad inf.*

Caſ. 1. *If the Ordinate is contain'd under* QR^{π}, *and* R *does not divide* Q you'll have $\lambda - 1 = \pi$. & $R^{\lambda-1} = R^{\pi}$.

But if R divides Q once you'll have $\lambda - 1 = \pi + 1$ & $R^{\lambda-1} = R^{\pi+1}$.

twice, ... $\lambda - 2 = \pi + 2$ & $R^{\lambda-1} = R^{\pi+2}$.

thrice, ... $\lambda - 3 = \pi + 3$ & $R^{\lambda-1} = R^{\pi+3}$.

And ſo on.

Caſ. 2. *If the Ordinate be an Irreducible Rational Fraction, with a Denominator, compos'd of two or more Terms: You muſt reſolve the Denominator into all its Prime Diviſors, & if there be any ſingle Diviſor which has no other equal to it, the Curve cannot be ſquared: But if there are two or more Diviſors equal, you muſt reject one of them; and if there are other two or more equal, and unequal to the reſt, you muſt alſo reject one of them; and ſo in all other equal ones, if there are more: Then the Diviſor which is left, or the Product of the Diviſors which are left, if there are more than one, muſt be ſet down or put for* R, *and the Reciprocal of its Square* R^{-2} *for* $R^{\lambda-1}$, *except where that Product is a Square, or Cube, or Bi-quadrate, &c. in which Caſe you muſt ſet down its Root* R, *and the Index of its Power* 2 *or* 3 *or* 4 *taken Negarively for* λ; *and reduce the Ordinate to the Denominator* R^2 *or* R^3 *cr* R^4 *or* R^5 &c.

As if the Ordinate be $\frac{z^5+z^4-8z^3}{z^5+z^4-5z^3-z^2+8z-4}=\frac{P}{Q}$.

Becauſe this Fraction is irreducible, and the Diviſors of the Denominator are even, *viz.* $z-1$, $z-1$, $z-1$, & $z+2$, $z+2$, he rejects one of each, and ſets down the Product of the reſt $z-1$, $z-1$, $z+2=z^3-3z+2$ for R, whence you'll have $\frac{1}{z^6-6z^4+4z^3+9z^2-12z+4}=\frac{1}{R^2}=R^{-2}$ to be plac'd in room of $R^{\lambda-1}$, then he reduces the Ordinate to the Denominator R^2 or $R^{1-\lambda}$ *viz.*

$Q:P::z^6-6z^4+4z^3+9z^2-12z+4\,(=R^2):z^6-9z^4+8z^3\left(=\frac{PR^2}{Q}.\right)$

then $\frac{z^6-9z^4+8z^3}{Q}=z^3\times\overline{8-9z+z^3}\times\overline{2-3z+z^3}|^{-2}$

Whence $a=8$, &c.

Whence $z^\theta R^\lambda\times\frac{\frac{1}{n}a}{re}=z^4R^{-1}\times\frac{\frac{1}{1}8}{4\times2}=\frac{z^4}{R}=\frac{z^4}{z^3-3z+z}=$ to the Area. All the Terms in the Series vaniſhing, except the Firſt.

Laſtly. If the Ordinate be an Irreducible Fraction, and its Denominator be the Product of the Rational Factor Q, and the Irreducible Surd Factor R^π, you muſt find all the prime Diviſors of its Root R, and reject one of each Magnitude, & multiply the Rational Factor Q, by the remaining Diviſors, if there are any; and if the Product is equal to the

Root R, or to any Power of that Root whoſe Root is an Integer, as ſuppoſe m, then will $\lambda - 1 = - \pi - m$, and $R^{\lambda - 1} = R^{-\pi - m}$. As if the Ordinate be $\dfrac{3q^5 - q^4x + 9q^3x^2 - q^2x^3 - 6qx^4}{qq - x^2\sqrt[3]{q^3 + q^2x - qx^2 - x^3}} = \dfrac{P}{QR^{\pi}}$. Now becauſe the Root R, of the Surd Factor $R = q^3 + q^2x - qx^2 - x^3$ has theſe Diviſors $q + x$, $q + x$, $q - x$ of two different Magnitudes, he rejects one of each Magnitude, and multiplies the Rational Factor $q^2 - x^2 = Q$, by the remaining Diviſor $q + x$; and becauſe the Product $q^3 + q^2x - qx^2 - x^3 = R$ he makes $m = 1$. And then, ſince π is $\frac{1}{3}$, $\lambda - 1$ becomes $= - \frac{4}{3}$. He reduces therefore the Ordinate to the Denominator $R^{-\frac{4}{3}}$ and you have $z^0 \times 3qx^6 + 2q^5x + 8q^4x^2 + 8q^3x^3 - 7q^2x^4 - 6x^5 \times \overline{q^3 + q^2x - qx^2 - x^3}\big|^{-\frac{4}{3}}$.

Whence defining the Coefficients by Comparing them with thoſe of the General Ordinate you'll have $3q^6 + 2q^5x + 8q^4x^2$, &c. $= a + bz^n + cz^{2n}$ &c. and $\overline{q^3 + q^2x - qx^2 - x^3}\big|^{-\frac{4}{3}} = \overline{e + fz^n + gz^{2n}, \text{\&c.}}\big|^{\lambda - 1}$. Whence $a = 3q^6$. $b = 2q^5$, &c. $e = q^3$. $f = qq$, &c.

$\theta - 1 = 0$. $\theta = 1 = n$. $\lambda = - \frac{1}{3}$. $r = 1$. $s = \frac{2}{4}$. $t = \frac{1}{3}$. $v = 0$. &c. &c. All the Terms in the Series after the third vaniſhing.

Sir *Iſaac* adds another Propoſition, which is a further Step in a Series of Propoſitions of the ſame Nature aſcending ad Infinitum. Some Years after theſe things were communicated to ſeveral of his Friends, as it is evident by Dr. *Wallis*, *pag.* 391 of the laſt Edition: Dr. *David Gregory*, Profeſſor of Mathematicks in the Univerſity of *Edinburgh*, and afterwards at *Oxford*, found out another Series of the ſame kind with that of Sir *Iſaac Newton*'s for Binomial Curves, but involv'd in more Intricate and Complex Terms; which Dr. *A. Pitcarn* Printed three or four Days before they had ſeen Sir *Iſaac Newton*'s, and he ſubjoins four Examples, which the Reader may pleaſe to conſult at leiſure, and he there alſo aſſerts, that the ſame Method might be applied to Infinitinomial Curves.

CHAP. VII.

Of the Fluents of Logarithmical and Exponential Fluxions, &c.

THE Invention of *Exponential Fluxions*, and their Affinity with and Relation to Logarithmical Curves is claim'd by two Gentlemen, *viz. Leibnitz* (as appears in thoſe Letters publiſh'd by Dr. *Wallis* Vol. 3.) and Mr. *John Bernoulli*. In the Year 1697, in the Month of *March* (upon an Occaſion offer'd by Mr. *Nieuwentitiit*) Mr. *Bernoulli* gives this Relation of his own Invention, and lays down the following Principles of it.

He ſays that this Invention is on this Account to be attributed to him, in as much as he had found it out five Years before, without having receiv'd any hint of it from Mr. *Leibnitz*, as alſo *Leibnitz* himſelf confeſſes in the *Act. Eruditor.* of the Year 1695, *pag.* 314.

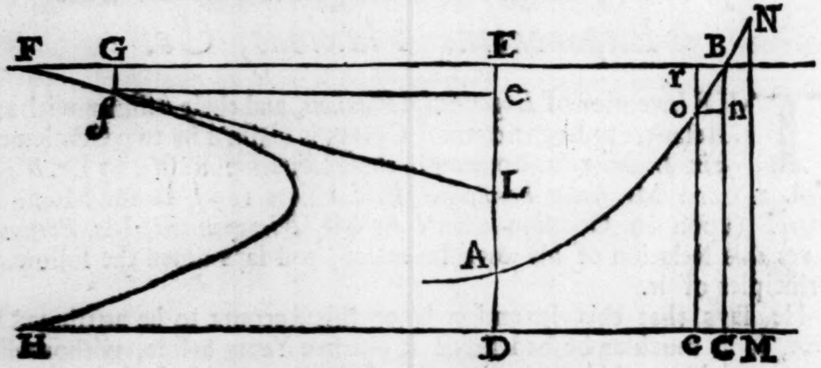
Mr. *Bernoulli* alſo in his turn, owns that Mr. *Leibnitz* did likewiſe invent it. *I us'd to call* (ſays he) *a Quantity rais'd to an indeterminate Power (before ever I knew any thing of Mr.* Leibnitz's *having conſider'd the Matter) by the Name of* Percurrent, *becauſe it runs thro' all poſſible Dimenſions, hence I gave alſo the ſame Name to Æquations conſiſting of ſuch Quantities, as alſo to Curves denoted or expreſs'd by them; but becauſe the Name of* Exponential *was made uſe of by that excellent Geometer, in honour to him, I reſolv'd alſo to make uſe of the ſame, and diſmiſs the former, leſt diverſity of Names might breed ſome Confuſion to the Reader, as to the Thing itſelf. I therefore conceiv'd, an Exponential Quantity, as a Mean, between a common Algebraic and Tranſcendent one; for it reſembles an Algebraic one, as being expreſs'd in Finite, tho' Indeterminate Terms; and a Tranſcendent one, becauſe it cannot be exhibited by any Algebraic Conſtruction. The Logarithmical Line (the moſt ſimple of common Tranſcendent ones) is of this Number of Curves, and all other Exponential ones may be conſtructed by means of this, and their Tangents determin'd as ſhall afterwards appear.* Then (for the Sake and Inſtruction of the aforeſaid *Nieu-*

ventiit) he there ſhews (what before was ſufficiently known) theſe three Things, *viz.*

1. How one and the ſame Indeterminate Line may, in the Logarithmical Curve be one of the Ordinates, and at the ſame time the Abſciſſe of another Curve.

2. How from this Equation $x^v = y$, you may infer this $vlx = ly$, where, by lx, ly, lz, *&c.* he underſtands the Logarithms of x, y, z, *&c.*

3. How lx may be the Integral (or Fluent) of $\frac{\dot{x}}{x}$.



1. Let DE be the Ax of the Curve HGF, & DC perpendicular to it, the Ax of the Logarithmical Line AB; produce the Ordinate FE till it meet the Logarithmical Line AB in B, whence having let fall the Perpendicular BC it will be equal to the Abſciſſe DE; but Bb is an Ordinate of the Logarithmical Curve; it is therefore Evident how the Abſciſſes DE of any Curve HGF may be Ordinates, not only in the Logarithmical Curve, but alſo in any other Curve AB, and that ſo as the Relation may always be preſerv'd between DE and EF, howſoever DE increaſes, whether equally or unequally, ſo that notwithſtanding taking the Differences of the Ax of the Logarithmical Curve Cc equal, the Differential Quantity FG will always preſerve its general relation to the Differential Quantities Ec, or FE to the Sub-tangent EL.

2. To demonſtrate the ſecond Thing propos'd, *viz.* that $vlx = ly$ you have only to inſpect the common Tables of Logarithms, which ſhew that the Logarithm of a Square, or Cube, or Biquadratick Number, *&c.* is Double, Triple, Quadruple of the Logarithm of the Root, whence univerſally the Log. $x^v = vlx$, *&c.* it remains to prove,

The Third, which I do thus; if $AD = 1 =$ to the Subtangent and

" BC = to the Number, then will, as is evident, DC be its Logarithm, " but by the Nature of the Logarithmical Line, the Subtangent in " $Bn = BC \times Cc$, and conſequently $\frac{Bn}{BC} = Cc$, hence if BC is x, y, z, &c. " and DC lx, ly, lz, &c. then will $\frac{\dot{x}}{x}, \frac{\dot{y}}{y}, \frac{\dot{z}}{z}$, &c. $= \varphi : lx, ly, lz$, &c. " Therefore alſo the $F : \frac{\dot{x}}{x}, \frac{\dot{y}}{y}, \frac{\dot{z}}{z} = lx, ly, lz$, &c. whence &c. and " the Demonſtration is general, and limited by no Condition, nor in " any Reſpect. For we have demonſtrated that in what Reaſon ſoever " DC, DE, or *bc*, BC increaſe; or by whatever Name they are diſtin- " guiſh'd, whether by *x* or *y*, whether they are conſider'd abſolutely or " relatively, that their Logarithms are D*c*, DC. Now that AD is " aſſum'd = 1 = to the Subtangent, that is only done for Compen- " dium, or Shortneſs ſake; for you may ſnbſtitute any Quantity what- " ever, and conſequently the Subtangent, for Unity; only taking this " care, that afterwards you refer to, or meaſure all other of the Lines, " by that aſſum'd one. Having now, I think, ſufficiently diſcuſs'd all " Difficulties, I come to treat down right of the Buſineſs itſelf, *viz.* " of Exponentials. Exponential Quantities are of Diverſe Kinds or " Degrees, the loweſt whereof (which hitherto has only been conſi- " der'd) is when the Exponent conſiſts of ordinary Indeterminates as " y^m, x^n, z^p, where $^{m,\ n,\ p}$ may be ſimply Indeterminate Quautities.

" An Exponential Quantity of the Second Degree, is when the Expo- " nent itſelf is an Exponential Quantity as y^{m^n}; and univerſally an Ex- " ponential Quantity of any Degree has for its Exponent an Exponential " Quantity of the next ſuperior Degree. You are to underſtand the " ſame in referring Exponential Æquations and Curves to their proper " Degrees. You are alſo to take Notice, that if an Æquation conſiſts " of different Exponentials of different Degrees, then the Æquation as " alſo the Curve denoted by it, ſhall be denominated *a potiori* or from " the higheſt Fluxion; in relation to all which, take what follows.

" Suppoſe any Logarithmical Curve as AB, taking the Subtan- " gent for Unity, and the Ordinate AD equal to it; let the Ordi- " nate B*b*, any way variable, whether it be an Ordinate or an Abſciſſe " of any other Curve, ſo it increaſe [or decreaſe] equally or unequally, " *per minima* (*or by Indefinitely ſmall Parts*) be denoted by x, y, or z, " and conſequently $DC = lx, ly, lz$. $Bn = \dot{x}, \dot{y}, \dot{z}$. $Cc = \varphi : lx, ly, lz$, &c.

“ Whence you have this general Rule, *The Fluxion of a Logarithm,*
however Compounded, is equal to the Fluxion, of a Number, divided by a
“ *Number* as $\varphi : l : \sqrt{xx+yy} = \frac{x\dot{x}+y\dot{y}}{xx+yy}$. To find therefore the Fluxion
“ of an Exponential Quantity of the firſt Degree m^n make $m^n = t$.
“ Therefore $nlm = lt$; but by the general Rule $\varphi : lm = \frac{\dot{m}}{m}$ and $\varphi : lt$
“ $= \frac{\dot{t}}{t}$ and therefore $\frac{n\dot{m}}{m} + lm\dot{n} = \frac{\dot{t}}{t} =$ (by reaſon of $m^n = t$) $\frac{\dot{t}}{m^n}$;
“ whence $\dot{t}$ or $\varphi : m^n = nm^{n-1}\dot{m} + m^n lm\dot{n}$; which ſuggeſts a firſt Special
“ Rule for Exponential Quantities of the firſt Degree, for you may
“ take m and n for Quanties any how compos’d of Indeterminate ones.
“ Suppoſe now an Exponential Quantity of the ſecond Degree m^{n^p}
“ make it equal to t, and conſequently $n^p lmn = lt$, and taking the
“ Fluxions on both ſides $n^p \varphi : l, m + l, m \varphi : n^p = \varphi : l,t$. But be-
“ cauſe by the firſt Special Rule $\varphi : l,m = \frac{\dot{m}}{m}$ and $\varphi : l,t = \frac{\dot{t}}{t}$ you’ll
“ have the ſecond Rule for Exponentials of the ſecond Gender, which
“ is this $\varphi : m^{n^p} = n^p m^{n^p - 1}\dot{m} + pn^{p-1} m^{n^p} lm\dot{n} + n^p m^{n^p} l,ml,n\dot{p}$. You may
“ after the ſame way find the following Rules for Exponentials of
“ higher Genders, nor is it more difficult to find the Fluxions of
“ Quantities however compounded from them, as $\varphi : m^n p^q = p^q \times \varphi : m^n$
“ $+ m^n \times \varphi : p^q$ where if you ſubſtitute the Value of $\varphi : p^q$, $\varphi : m^n$ found
“ as before, you’ll have the Fluxion ſought.

The Examples he gives are as follow.

“ 1. To conſtruct a Curve whoſe Nature is expreſs’d by this Exponential
“ Æquation $x^x = y$; to determine its Tangent, taking its Logarithms,
“ you have $xlx = ly$, drawing therefore a Logarithmic Line
“ AB whereof the Subtangent ſhall be $= 1 =$ AD, and let DE or
“ BC $= x$, then will DC $= lx$; and conſequently, if you make AD :
“ BC :: DC : DM, DM will be $= ly$ & MN $= y$, to which, if you
“ draw

“ draw equal EF, an Ordinate to the Abſciſſe DE, the Point F will be
“ in the required Curve HGF, the Tangent in the Point F is thus deter-
“ min'd by the firſt Special Rule, $\dot{y} = x^x\dot{x} + x^x l : x\dot{x} =$ (ſubſtituting y
“ in the Room of x^x) $y\dot{x} + ylx\dot{x}$ that is $y + ylx : 1 :: \dot{y} : \dot{y}$: ſubtang. $\frac{1}{1 + lx}$
“ hence taking EL a third proportional to AD + DC and AD, FL will
“ be the Tangent; which we have therefore expreſs'd in ordinary, or
“ or common Terms.

“ Under theſe Terms *Ordinary* and *Aſſignable*, Mr. *Leibnitz* compre-
“ hends, not only an Algebraical, but alſo a Tranſcendent Quantity, ſo
“ it be a finite one; as the Periphery of a Circle cannot be expreſs'd in
“ Algebraic Terms, is it therefore not an ordinary or finite Quantity?
“ or is it leſs Aſſignable, when it may be aſſign'd by one turn of the
“ Compaſs?

“ 2. If there be propos'd a Curve, whoſe Æquation is $x^y = a$; you'll
“ have, when it is reduc'd to Logarithms $ylx = la$, whence, as before
“ we have ſhewn by the Logarithmical Line, you'll have the Conſtruct-
“ ion of the Curve; and finding the Fluxions by the firſt Special Rule,
“ you have $\dot{x} + lx\dot{x} = ly\dot{y}$, whence alſo flows the Determination of the
“ Subtangent; here it may be worth while to Obſerve that the Quadra-
“ ture of this Figure may be exhibited without a Series, for the
“ F : $y\dot{x} = \frac{2xxlx - xx}{la}$.

“ 3. If there was propos'd $a^x = y$; and conſequently $xla = ly$, there
“ would come out $yla\dot{x} = \dot{y}$; that is, the Subtangent would be equal
$\frac{1}{la}$ = to a conſtant Quantity; whence may be concluded that the
“ Æquation propos'd, expreſs'd the Logarthmical Curve. Wherefore
“ by one Stroke of the Pen, I have ſhewn a plain and eaſy Way (as
“ Mr. *Niewentiit* deſires) of reducing the Logarithmical Line, known
“ now to every Body, to an Exponential Æquation.

“ 4. Suppoſe now an Exponential Æquation of ſeveral Terms, *viz.*
“ $x^x + x^a = x^y + y$; taking the Fluxions on both ſides of each Term
“ by the firſt Special Rule, you'll have $\varphi : x^x = x^x\dot{x} + x^x lx\dot{x} . \varphi : x^a =$
“ $ax^{a-1}\dot{x} . \varphi : x^y = yx^{y-1}\dot{x} + x^y lx\dot{y}$ which being diſpos'd in due Order
“ will give this Æquation $x^x\dot{x} + x^x x\dot{x} + ay^{a-1}\dot{x} - yx^{y-1}\dot{x} = x^y lx\dot{y} + \dot{y}$;
“ making

" making therefore as $\dot{y}$ to $\dot{z}$; that is as $x^x + x^x lx + cx^{c-1} - yx^{y-1}$ to " $x^x lx + 1$, ſo y to a Fourth, and this Fourth will be the Subtangent " ſought of the propoſed Curve. Take Notice, in the mean while, " that the Multitude of Terms no ways hinders the Contraction of " them, for ſince, by means of a Logarithmical Line, there may be " aſſign'd a Quantity equal to each, it is evident, that thoſe Quantities " thus aſſign'd, and taken together, will form the Quantity ſought.

" The Examples here already brought, will abundantly Illuſtrate the " Proceſs to be made uſe of in other Exponentials of the firſt Gender, " and from them it will not be difficult to ſee which way the Method " is to be apply'd to higher Genders; for in all you are to obſerve the " ſame Method of Operation; ſo far Mr. *John Bernoulli*.

Dr. *Cheyne* has ſhewn Analytically, and by a Demonſtration drawn from the Nature of Logarithms, the Foundation of theſe Methods as may be ſeen pag. 14. of his *Inverſe Method*, and has carried the Method of finding the Fluents of Logarithmical Fluxions very high. Which before we enter upon, we'll ſhew you from him, *viz.* from pag. 14 and 28. what thoſe Fluxions are in the following ſhort *Synopſis*.

$$\Phi : \left\{ \begin{array}{l} lx = \frac{\dot{x}}{x} \\ lx + 1 = \frac{\dot{x}}{x+1} \\ l : lx = x^{-1} l^{-1} x\dot{x} = \frac{\dot{x}}{xlx}. \\ l^n x = nx^{-1} l^{n-1} x\dot{x} \\ x^m l^n : lx = mx^{m-1} l^n : lx\dot{x} + nx^{m-1} l^{n-1} : lx l^{-1} x\dot{x}. \\ x^m l^n x l^r : lx + a^{m+n+r} = \left\{ \begin{array}{l} mx^{m-1} l^n x l^r : lx\dot{x} + \\ nx^{m-1} l^{n-1} x l^r : lx\dot{x} + \\ rx^{m-1} l^{n-1} x l^{r-1} : lx\dot{x} \text{ \&c. \&c.} \end{array} \right. \end{array} \right.$$

Or that the Correſpondent Fluents are the proper ones of thoſe Fluxions, or, which is all one, proportional to 'em.

Hence Dr. *Cheyne* exhibits the Fluents of Logarithmical Fluxions in the more general Caſes; an Explication whereof we ſhall here ſhew for the ſake of Beginners, and of ſome that have made ſome Scruples about them; premiſing firſt of all the Rule for Aſſumptions ad Inf. as laid down by Dr. *Cheyne*, which is this

$\overline{\overline{\overline{x^{m+1} + lx + l : lx + l :. l : lx + 1}|^{n+1}}|^{p+1}}|^{q+1}$ &c. which being evolv'd and adfected with Indeterminate Coefficients, will give the Series to be aſſum'd for the Logarithmical Fluents ſought; and thence you have a Progreſſion of Aſſumptions ad Inf. as ſhall appear from what follows. Noting in the mean while, (that tho' the Aſſumption may reach only to ſome Terms, whoſe Indices may go on in an Arithmetical Progreſſion from 0 to Infinite) that the Fluents of all of the like Nature, may be exhibited ad Infinitum, or generally, you muſt make uſe of an Infinite Progreſſion of Terms with Indeterminate Indices, under which all will be comprehended; and from the Primary aſſumption (as we have heretofore hinted) all the reſt, tho' Infinitely Infinite, will come out, as ſhall be more largely ſhewn further on. Now 1ſt, Let the given Fluxion be $x^m l x \dot{x}$ (the ſame with that of Dr. *Cheyne*, pag. 14.) Now, according to the Rule above-mention'd, you muſt aſſume $x^{m+1} \times \overline{lx + 1}|$ adfected with Indetermin'd Coefficients, *viz.* $Bx^{m+1}lx + Cx^{m+1}$ which will yet thus further appear from the Nature of the direct Method: Aſſume firſt of all $Bx^{m+1}lx$; and having taken its Fluxion, you'll have $\overline{m+1} \times Bx^m lx\dot{x} + (Bx^{m+1}\frac{\dot{x}}{x} =)\ Bx^m\dot{x}$; Now compare the firſt Member with the Fluxion given, which having done and determin'd the Coefficient $B = \frac{1}{m+1}$, there will remain the other part of the Fluxion, *viz.* $Bx^m\dot{x}$, for which aſſume the Fluent Cx^{m+1}, and ſo the Aſſumption is terminated or wholly exhauſted, and when the Primary Aſſumption is Infinite, there will (according to the Nature of the Fluxion) remain an Infinite Series of Remaining Fluxions, for which another Infinite Series muſt be aſſum'd, and ſo on, till the Fluent runs into ſeveral, or Infinite Series; and this flows immediately from the thing itſelf, the way as above ſhewn. Thus in Dr. *Cheyne*, aſſuming for the Fluxion $x^m l^n x\dot{x}$ only the firſt or Primary Fluent, that alone will give all the ſubſequent Terms and Series.

1. The

1. The Fluent of the Fluxion $rx^m l^n x\dot{x}$ will be abſolv'd or finiſh'd in one Infinite Series, and give $\frac{r}{m+1}x^{m+1}l^n x - \frac{nr}{\overline{m+1}|^2}x^{m+1}l^{n-1}x$ &c. ad Inf. as will appear from the Fluxion of the aſſum'd Fluent, and its Compariſon with the given one; where if $n =$ to any Integer affirmative Number, or to 0 in any particular Fluxion, which falls under this Formula, the Series will break off, and the Fluent be exhibited in Finite Terms.

2. If the given Fluxion was $x^m l:lx\dot{x}$, you muſt aſſume $x^{m+1} \times \overline{l:lx \text{ evolv'd \&c.}}|$ which in this ſort of Evolution becomes $l^{-1}x + l^{-2}x + l^{-3}x$ &c. ad Inf. wherefore you muſt aſſume $x^{m+1} \times \overline{l:lx + l^{-1}x}$ &c. ad Inf. adfected with Indetermin'd Coefficients, *viz.* $Ax^{m+1}l:lx \ldots Bx^{m+1}l^{-1}x \ldots Cx^{m+1}l^{-2}x$ &c.

Suppoſe now 3ly, the given Fluxion to be $x^m l^2:lx\dot{x}$; its primary aſſum'd Fluent, *viz.* $Ax^{m+1}l^2:lx \ldots Bx^{m+1}l^{-1}xl:lx \ldots Cx^{m+1}l^{-2}xl:lx$ &c. will leave for the ſecond Series to be aſſum'd, this Series of remaining Fluxions, *viz.* $x^m l^{-2}x\dot{x} \ldots x^m l^{-3}x\dot{x}$ &c. for which you muſt again aſſume $x^{m+1}l^{-2}x \ldots x^{m+1} \times l^{-3}x$ &c. ad Inf. adfected with Indeterminate Coefficients, and the Series will be exhauſted, or there will be found no farther remaining Fluxions, &c.

Again, if you had the given Fluxion $x^m l^3:lx\dot{x}$ from the primary aſſum'd Fluxion (adfected with Indetermin'd Coefficients)

viz. $x^{m+1}l^3:lx \ldots l^{-1}xl^2:lx \ldots l^{-2}xl^2:lx \ldots l^{-3}xl^2:lx$ &c.
There would come out $\ldots\ldots\ldots l^{-2}xl:lx \ldots\ldots l^{-3}xl:lx \ldots l^{-4}l:lx$ &c.
$\ldots\ldots\ldots l^{-3}x \ldots\ldots\ldots l^{-4}x \ldots\ldots\ldots l^{-5}x$ &c.

Which muſt alſo be adfected with Indetermin'd Coefficients, and ſo would be exhauſted; and ſo on. Whence univerſally in that Kind, if there was given $x^m l^n:lx\dot{x}$; you muſt aſſume *
$x^{m+1}l^n:lx \ldots, x^{m+1}l^{-1}xl^{n-1}:lx \ldots x^{m+1}l^{-2}xl^{n-1}:lx$ &c. and thence you'll have $x^{m+1}l^{-2}xl^{n-2}:lx \ldots x^{m+1}l^{-3}xl^{n-2}:lx$ &c.
&c. ad Inf. &c. ad Inf. adfected

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* *N.B.* From the firſt Term alone aſſum'd, the reſt will come out, firſt by the Direct, and then the Inverſe Method.

adfected with Indetermin'd Coefficients, *viz.*

$$\alpha x^{m+1}l^{n}:lx\,..\,Ax^{m+1}l^{-1}xl^{n-1}:lx\,..\,Bx^{m+1}l^{-2}xl^{n-1}:lx\,..\,Cx^{m+1}l^{-3}xl^{n-1}:lx \text{ \&c.}$$

$$Dx^{m+1}l^{-2}xl^{n-2}:lx\,..\,Ex^{m+1}l^{-3}xl^{n-2}:lx\,..\,Fx^{m+1}l^{-4}xl^{n-2}:lx \text{ \&c.}$$

&c. &c. &c.

And having taken the Fluxions, you'll have $x^{m}l^{n}:lx\dot{x}=\overline{m+1}\alpha x^{m}l^{n}:lx\dot{x}$. Whence 1°. $1=\overline{m+1}\alpha$ & $\frac{1}{\overline{m+1}|}=\alpha$ &c. And going on you'll have for the remaining Series of Comparisons

$$\left.\begin{array}{r} n\alpha x^{m}l^{-1}xl^{n-1}:lx\dot{x} \\ m+1Ax^{m}l^{-1}xl^{n-1}:lx\dot{x} \end{array}\right\}\left.\begin{array}{l} -1Ax^{m}l^{-2}xl^{n-1}:lx\dot{x} \\ m+1B\ldots\ldots\ldots\ldots \end{array}\right\}\begin{array}{l} -2Bx^{m}l^{-3}xl^{n-1}:lx\dot{x} \\ m+1C\ldots\ldots\ldots\ldots \end{array}$$

$$\left.\begin{array}{r} -3Cx^{m}l^{-4}xl^{n-1}:lx\dot{x} \\ m+1D\ldots\ldots\ldots\ldots \end{array}\right\}\text{\&c. ad Inf.}$$ Whence there would come out

2°. $n\alpha=m+1A$. $A=\frac{n\alpha}{m+1}=\frac{nr}{\overline{m+1}|^{2}}$.

3°. $-1A=m+1B$. $B=-\frac{1A}{m+1}=-\frac{1nr}{\overline{m+1}|^{3}}$.

4°. $-2B=m+1C$. $C=-\frac{2B}{m+1}=-\frac{1\times2nr}{\overline{m+4}|^{4}}$.

5°. $-3C=m+1D$. $D=-\frac{3C}{m+1}=-\frac{1\times2\times3nr}{\overline{m+1}|^{5}}$.

6°. $-4D=m+1E$. Whence $E=-\frac{4D}{m+1}=-\frac{1\times2\times3\times4nr}{\overline{m+1}|^{6}}$.

and so ad Inf. and for a farther Comparison there would remain

$$\overline{n-1}Ax^{m}l^{-2}xl^{n-2}:lx\dot{x}\ldots\ldots\overline{n-1}Bx^{m}l^{-3}xl^{n-2}:lx\dot{x}\ldots\ldots$$

$$\overline{n-1}Cx^{m}l^{-4}xl^{n-2}:lx\dot{x}\ldots\ldots\overline{n-1}Dx^{m}l^{-5}xl^{n-2}:lx\dot{x} \text{ \&c.}$$

For which you muſt aſſume

$$Ex^{m+1}l^{-2}xl^{n-2}:lx\ldots Fx^{m+1}l^{-3}xl^{n-2}:lx\ldots Gx^{m+1}l^{-4}xl^{n-2}:lx \text{ \&c.}$$

Whereof having taken the Fluxion, there will come out another Series of remaining Fluxions, *viz.*

$$\left.\begin{array}{c} m+1Ex^{m}l^{-2}xl^{n-2}x\dot{x}-2Ex^{m}l^{-3}xl^{n-2}:lx\dot{x} \\ m+1F\ldots\ldots\ldots\ldots \end{array}\right\} \begin{array}{c} +\overline{n-2}\times Ex^{m}l^{-3}xl^{n-3}:lx\dot{x}\ \&c. \\ \&c. \end{array}$$

Which would farther remain for a ſubſequent Series of Aſſumptions, and ſo ad Inf. and you'll have $n-1A=m+1E$. $E=\frac{n-1A}{m+1}$. $-2E+mF=\overline{n-1}B$. or $m+1F=\overline{n-1}B+2E$. & $F=\frac{\overline{n-1}B+2E}{m+1}$ &c. which coincide with thoſe of Dr. *Cheyne.* And hence you have the Natural origine of Infinitely-Infinite Series.

Suppoſe again, this given Fluxion $px^{m}l^{r}xl^{n}:lx\dot{x}$ *viz.* the Example of Dr. *Cheyne* pag. 32. aſſume (or let the firſt Term of the Primary Aſſumption) be 1°. $ax^{m+1}l^{r}xl^{n}:lx$ its Fluxion will be

$\overline{m+1}ax^{m}l^{r}xl^{n}:lx\dot{x}+rax^{m}l^{r-1}xl^{n}:lx\dot{x}+nax^{m}l^{r-1}xl^{n-1}:lx\dot{x}$ Wherefore $px^{m}l^{r}xl^{n}:lx\dot{x}=\overline{m+1}ax^{m}l^{r}xl^{n}:lx\dot{x}$: Whence $p=\overline{m+1}a$. $a=\frac{p}{m+1|}$.

But becauſe there now remain $rax^{m}l^{r-1}xl^{n}:lx\dot{x}+nax^{m}l^{r-1}xl^{n-1}:lx\dot{x}$.

2. Aſſume $Ax^{m+1}l^{r-1}xl^{n}:lx=$ to the ſecond Term of the Fluent whoſe Fluxion will be

$m+1Ax^{m}l^{r-1}xl^{n}:lx\dot{x}-\overline{r-1}Ax^{m}l^{r-2}xl^{n}:lx\dot{x}+nAx^{m}l^{r-1}xl^{n-1}:lx\dot{x}$. And the next Compariſon will be $m+1Ax^{m}l^{r-1}xl^{n}:lx\dot{x}=rax^{m}l^{r-1}xl^{n}:lx\dot{x}$. Or $m+1A=ra$. And $A=\frac{ra}{m+1}=\frac{rp}{m+1|^{2}}$.

Whence $Ax^{m+1}l^{r-1}xl^{n}:lx=\frac{rp}{m+1|^{2}}x^{m+1}l^{r-1}xl^{n}:lx$.

3. Now for the next Term of the remaining Fluxion aſſume $Bx^{m+1}l^{r-2}xl^{n}:lx$; and its Fluxion will be

$\overline{m+1}Bx^{m}l^{r-2}xl^{n}:lx\dot{x}+\overline{r-2}Bx^{m}l^{r-3}xl^{n}:lx\dot{x}+nBx^{m}l^{r-2}xl^{n-1}:lx\ddot{x}$.

And$-\overline{r-1}Ax^m l^{r-2}xl^n : lx\dot{x} = \overline{m+1}Bx^m l^{r-2}xl^n : lx\dot{x}$

And$-\overline{r-1}A = \overline{m+1}B. -\frac{\overline{r-1}A}{m+1} = B = \frac{\overline{r-1}\times r\times p}{\overline{m+1}|^3}$.

4°. Aſſume $Cx^{m+1}l^{r-3}xl^n : lx$ whoſe Fluxion will be

$\overline{m+1}Cx^m l^{r-3}xl^n : lx\dot{x} + |\underline{\overline{r-3}Cx^m l^{r-4}xl^n : lx\dot{x} + nCx^m l^{r-3}xl^{n-1} : lx\dot{x}\quad}$

And $\overline{r-2}B$ will $= \overline{m+1}C$. & $C = \frac{\overline{r-2}B}{m+1} = \frac{\overline{r-2}\times\overline{r-1}\times r\times p}{\overline{m+1}|^4}$.

5°. Aſſume $Dx^{m+1}l^{r-4}xl^n : lx$. and you'll have

$\overline{m+1}Dx^m l^{r-4}xl^n : lx\dot{x}|\underline{+\overline{r-4}Dx^m l^{r-1}xl^n : lx\dot{x} + nDx^m l^{r-4}xl^{n-1} : lx\dot{x}\quad}$

& you'll have $\overline{r-3}C = \overline{m+1}D. \frac{\overline{r-3}C}{m+1} = D = \frac{\overline{r-3}\times\overline{r-2}\times\overline{r-1}\times r\times p}{\overline{m+1}|^5}$.&c.

ad Inf. for the firſt Series; whence from the third Remainders of the primary Aſſumption you'll have a Series of Fluxions, which will give you, or determine, the ſecond Series to be aſſum'd, and ſo onward ad Inf. *viz.*

for $n\alpha x^m l^{r-1}xl^{n-1} : lx\dot{x} \ldots nAx^m l^{r-1}xl^{n-1} : lx\dot{x} \ldots nBx^m l^{r-2}xl^{n-1} : lx\dot{x} \ldots\ldots$

$\ldots nCx^m l^{r-3}xl^{n-1} : lx\dot{x} \ldots\ldots nDx^m l^{r-4}xl^{n-1} : lx\dot{x}$ &c. You muſt aſſmue

$Gx^{m+1}l^{r-1}xl^{n-1} : lx + Hx^{m+1}\times l^{r-2}xl^{n-1} : lx$ &c. ad Inf. which will yet leave another remaining Series, and ſo ad Inf. and the Fluxion of this laſt Equation is to be compared with its reſpective Series of remaining Quantities, and there will yet remain another Series, *viz.*

$\varphi : Gx^{m+1}l^{r-1} : xl^{n-1} : lx = \overline{m+1}Gx^m l^{r-1}xl^{n-1} : lx\dot{x} \ldots\ldots\ldots\ldots$

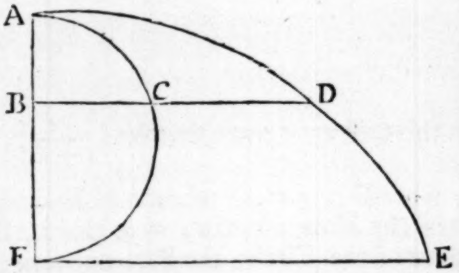
$\ldots\ldots \overline{r-1}Gx^m l^{r-2}xl^{n-1} : lx\dot{x} \ldots\ldots \overline{n-1}Gx^m l^{r-2}xl^{n-2} : lx\dot{x}$ and now you'd have $n\alpha = \overline{m+1}G$. or $G = \frac{n\alpha}{m+1} = \frac{np}{\overline{m+1}|^2}$, and ſo on ad Inf.

Dr. *Cheyne* applies theſe Series to the finding the Fluents of Logarithmico-Logarithmical Fluxions, and ſhews their Uſe in Exponentials of the ſecond Gender; as may be ſeen *Pag.* 34. of his *Inverſe Method*, which the Reader may, if he pleaſes, conſult. Now if there were other Degrees of Irrationality, theſe Infinitely-Infinite Series might be collected into one Partition, wherein each would make one Line, and ſo go Infinitely to the Right; and the others being ſubjoin'd Infinitely downwards, higher Degrees would make 2, 3, 4, *&c.* ſuch Infinitely-Infinite Series, and ſo onwards ad Inf.

CHAP. VIII.

Of Mr. Craig's *Method of Squaring Irrational Curves,* &c.

MR. *Craig*, after having long before publiſhed ſeveral Quadratures of Curves, gives us a Treatiſe of his Method of handling Irrational ones, as may be ſeen by his Specimen in the *Philoſophical Tranſactions*, Nº. 232, where you'll find as follows: " Let " ACF be a Semicircle, where- " of the Diameter is AF, and " ADE a Curve Geometri- " cally Irrational, whoſe Or- " dinate BD cuts the Semi- " circle in C; and let us thus " denote the following Quan- " tities, *viz.* Make the Dia- " meter AF $= 2a$, the Abſciſſe " AB $= y$, the Arch AC $= v$, " the Ordinate BD $= z$; and " let ruy^n be a general Equation expreſſive of the Nature of Geo- " metrically Irrational Curves, as ADE, where r denotes any given " determinate Quantity, and n an Indefinite Index of the Indetermi- " nate Quantity y. I ſay, the Area ABD will be $= \frac{ruy^{n+1}}{n+1} - qv$



" $+ \sqrt{2ay - yy} \times \frac{ra}{\overline{n+1}|^2} y^n + \frac{2nra^2 + ra^2}{n \times \overline{n+1}|^2} y^{n-1} + \frac{aA \times \overline{2n-1}}{n-1} y^{n-2}$

" $+ aB$

“ $+\frac{aB\times 2n-3^{\,2}}{n-2}y^{n-2}+\frac{aC\times 2n-5}{n-3}y^{n-4}+\frac{aD\times 2n-7}{n-4}y^{n-5}$

“ $+\frac{aE\times 2n-9}{n-5}y^{n-6}+$, &c. In this Series you are in parricular to take
“ notice of what follows ; (1.) That the great Letters A,B,C, D,E,&c. de-
“ note the Coefficients of thoſe Terms that immediately precede them, *viz*.

$A=\frac{2nra^{2}+ra^{2}}{n\times n+1\times n-1}$, $B=\frac{aA\times 2n-1}{n-1}$, $C=\frac{aB\times 2n-3}{n-2}$, & ſic

“ and ſo on. (2.) That if the Exponent n be an Integer and poſitive Num-
“ ber, or equal to Nought; or if $2n$ be an odd Number, then the Qua-
“ drature of the Space, ABD will come out a Finite Quantity ; the
“ Series in ſuch Caſes breaking off. (3.) That q denotes the laſt or ab-
“ rumpent Term. (4.) That all thoſe Figures wherein the Series breaks
“ off, (or becomes Finite) have one Part Geometrically ſquareable, and
“ eaſily aſſignable by the Series : *viz.* if you take the Abſciſſe $y=\overline{r}|^{\frac{1}{n+1}}$
“ $\times\frac{}{\overline{nq+q}|}^{\frac{1}{n+1}}$; you'll find the Curve anſwering to this Abſciſſe to be
“ Geometrically ſquareable. (5.) That the Irrational Term is only to
“ be multiply'd into thoſe that follow it.

“ *Ex.* 1. Suppoſe $z=v$, becauſe in this Caſe $r=1$, $n=0$, therefore $\frac{ra}{\overline{n+1}|^{2}}y^{n}$
“ is the laſt or abrumpent Term, wherefore $q=a$, whence $ABD=vy$
“ $-av+a+\sqrt{2ay-yy}$: and then (by Note 4.) take $y=a$, that is, if
“ the Ordinate paſs thro' the Centre of the Circle, the Part anſwering
“ to it will be Geometrically ſquareable, *viz.* the Area $=aa$, i. e. to the
“ Square of the Radius.

“ *Ex.* 2. Suppoſe $z=\frac{vy}{a}$ now becauſe here $r=\frac{a}{1}$, $n=1$, therefore
“ $\frac{anraa+raa}{n\times\overline{n+1}|^{2}}y^{n-1}$ is the laſt or abrumpent Term, wherefore $q=\frac{3a}{4}$,
“ whence $ABD=\frac{vy^{3}}{2a}-\frac{3av}{4}+\frac{y+3a}{4}\sqrt{2ay-yy}$, and then if (by Note 4.)

“ you take $y=\sqrt{\frac{3aa}{2}}$, you'll have a Geometrically ſquareable Area a-“ greeing or correſponding to this Abſciſſe, *viz.* the Area $=\sqrt{\sqrt{6a^4}-\frac{3a^2}{2}}$ “ $\times\frac{\sqrt{3a^2}}{32}+\frac{3a}{4}$.

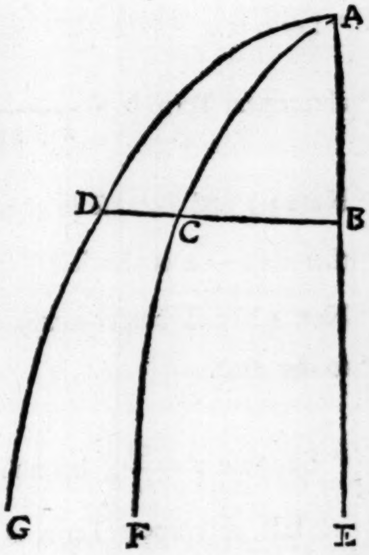
“ *Ex.* 3. Suppoſe $z=\frac{vy^2}{aa}$; in this Caſe $r=\frac{1}{aa}$, $n=2$, therefore “ $\frac{aA\times\overline{2n-1}}{n-1}y^{n-2}$ is the laſt abrumpent Term, therefore $q=\frac{5a}{6}$; whence “ by the Series you'll have ABD “ $=\frac{6vy^3-15a^3v+\overline{2ay^2+5a^2y+15a^3}\sqrt{2ay-yy}}{18a^2}$; Then if (by Not. “ 4.) you take $y=\sqrt[3]{\frac{5a^3}{2}}$ the Geometrically ſquareable Area correſpond-“ ing to this Abſciſſe will be $=\frac{2ay^2+5a^2y+15a^3}{18a}\times\sqrt{2ay-yy}$.

“ Secondly, Let ACB be a Parabola, whoſe Axis is AE, its Vertex A, “ and Latus Rectum BA.

“ And let ADG be a Geometrically “ Irrational Curve, whoſe Ordinate BD “ cuts the Parabola in C. Make the “ Abſciſſe AB $=y$, the Ordinate BD $=z$, “ the Parabolick Arch AC $=v$. And “ let the general Equation, expreſſing the “ Nature of Infinite Geometrically Irra-“ tional Curves, be $z=rvy^n$, wherein r “ denotes a given determinate Quantity, “ and n the Exponent of the indetermi-“ nate Quantity. I ſay, the Area ABD will

“ $=\frac{ry^{n+1}v}{n+1}-qv+\sqrt{2ay+yy}\times-$

“ $\frac{r}{n+2\times n+1}y^{n+1}-\frac{ra}{n+2\times\overline{n+1}|^2}y^n$



" $+ \frac{raa \times \overline{2n+1}}{n \times \overline{n+2} \times \overline{n+1}|^2} y^{n-1} - \frac{aA \times \overline{2n-1}}{n-1} y^{n-2} + \frac{aB \times \overline{2n-3}}{n-2} y^{n-3}$

" $\frac{aC \times \overline{2n-5}}{n-1} y^{n-4} +$, &c. In this Series you are to take Notice;

" 1. That the great Letters A, B, C, *&c.* denote the Coefficients of the " Terms that precede them. 2. That if the Exponent n be a poſitive " Integer Number, or equal to Nothing, or if $2n$ be an Odd Number, " then the Quadrature will be exhibited in a finite Number of Terms; " the Series in this Caſe breaking off.

3. " That $+q$ is equal to the laſt abrumpent Term.

4. " That of the Terms that are multiply'd by the Quantity $\sqrt{2ay+yy}$ " the laſt abrumpent one is to be doubled.

5. " That all thoſe Figures in which n is an Interger, Poſitive, and " Odd Number; or more generally, all thoſe Figures, wherein the laſt " abrumpent Term has an Affirmative Sign or $+$, have one Portion or " Part of them Geometrically ſquareable, and eaſily aſſignable by the " Series it ſelf, by taking the Abſciſſe as in Note 4. of the precedent " Series.

Examp. 1.

" Let $z = v$, now becauſe in this Caſe $r = 1$, $n = 0$, therefore the laſt " abrumpent Term is $- \frac{ra}{n + 2 \times \overline{n+1}|^2} y^n$, whence $+ q = - \frac{a}{2}$ (by " Note 3.) and becauſe in this Caſe $- \frac{a}{2}$ is the laſt abrumpent Term, " therefore $-a$ is the laſt Term to be multiply'd into $\sqrt{2ay+yy}$ (by " Not. 4.) and conſequently $vy + \frac{av}{2} + \sqrt{2ay+yy} \times \overline{-\frac{1}{2}y - a}. =$ " to the Area.

Examp. 2.

" Suppoſe $z = \frac{vy}{a}$, becauſe in this Caſe $r = \frac{1}{a}$, $n = 1$, therefore ' the laſt abrumpent Term is $- \frac{raa \times \overline{2n+1}}{n \times \overline{n+2} \times \overline{n+1}|^2} y^{n-1} = \frac{a}{4}$, whence

$q =$

" $q = \frac{a}{4}$, and $\frac{a}{2}$ is the laſt Term to be multiply'd into $\sqrt{2ay + yy}$;
" and conſequently $ABD = \frac{vyy}{2a} - \frac{av}{4} + \sqrt{2ay + y^2} \times -\frac{y}{6a} - \frac{y}{12}$
" $+ \frac{a}{2}$; and if you take $y = \sqrt{\frac{aa}{2}}$ the Area correſponding to it will be
" Geometrically ſquareable, *viz.* the Area $= \frac{1}{12}\sqrt{:\sqrt{2a^4} + \frac{a^2}{2}} \times$
" $5a - \sqrt{\frac{a^2}{2}}$.

" 3. Let ACF be a Semicircle; ADE a Curve Geome-" trically Irrational, the Ordinate whereof cuts the Semi-" circle in C; and let the Quantities remain noted as be-" fore, *viz.* the Diam. $AF = 2a$, the Abſciſſe $AB = y$, the Arch " $AC = v$, the Ordinate $BD = z$. And let $z = rv^2y^n$ be an Equation " expreſſing the Natures of the Curves ADE, wherein r denotes " any given and determinate Quantity, and n the indefinite Ex-" ponent of the indeterminate Quantity y. I ſay the Area will be

Tranſact. N° 235. *See Fig.* 1.

" $$ABD = \frac{rv^2y^{n+1}}{n+1} - qv^2 + v\sqrt{2ay - yy} \times \frac{2ra}{\overline{n+1}|^2}y^n$$

" $$+ \frac{2ra^2 \times \overline{2n+1}}{n \times \overline{n+1}|^2}y^{n-1} + \frac{aA \times \overline{2n-1}}{n-1}y^{n-2} + \frac{aB \times \overline{2n-3}}{n-2}y^{n-3}$$

" $$+ \frac{aC \times \overline{2n-5}}{n-3}y^{n-4} + \frac{aD \times \overline{2n-7}}{n-4}y^{n-5} + \frac{aE \times \overline{2n-9}}{n-5}y^{n-6}, \text{\&c.}$$

" $$- \frac{2ra^2}{\overline{n+1}|^3}y^{n+1} - \frac{2ra^3 \times \overline{2n+1}}{n^2 \times \overline{n+1}|^2}y^n - \frac{a^2A \times \overline{2n-1}}{\overline{n-1}|^2}y^{n-1} - \frac{a^2B \times \overline{2n-3}}{\overline{n-2}|^2}y^{n-2}$$

" $$- \frac{aC \times \overline{2n-5}}{\overline{n-3}|^2}y^{n-3}, \text{\&c.}$$

" In this Theorem you are to note; 1. That it is compos'd of two " infinite Series; the firſt whereof (joined by the Sign +) is multi-" ply'd into $v\sqrt{2ay - yy}$; but the Terms of the latter (adfected by the " Sign −) are abſolute. 2. That in the former Series the great Let-

" ters A, B, C, D, E, &c. denote the Coefficients of the Terms immedi-
" ately preceding them; and alſo denote the ſame Value in the latter
" they did in the former. 3. That the Quadrature is exhibited in a
" finite Quantity, when n is an Integer and Poſitive Number, or equal
" to Nought, or alſo when $2n$ is an Odd Number: For in theſe Caſes
" both Series will break off. 4. That $2q$ is equal to the laſt abrumpent
" Term of the former Series.

Examp. 1.

" Suppoſe $z = \frac{v^2}{a}$ becauſe in this Caſe $n = 1. r = \frac{1}{a}$ therefore the Area
" $= \frac{yv^2}{a} - v^2 + 2v\sqrt{2ay - yy} - 2ay$.

Coroll. " The whole Eigure is equal to twice a Square, whoſe Side is
" ACF, ſubſtracting the Square of the Diameter.

Examp. 2.

" Suppoſe $z = \frac{yv^2}{a^2}$ becauſe in this Caſe $n = 1, r = \frac{1}{a^2}$, therefore the Area
" ABD $= \frac{y^2v^2}{2a^2} - \frac{1}{4}v^2 + v\sqrt{2ay - yy} \times \overline{\frac{v}{2a} \times \frac{3}{2}} - \frac{1}{4}y^2 - \frac{3ay}{2}$.

Examp. 3.

" Suppoſe $z = \frac{y^2v^2}{a^3}$, becauſe in this Caſe $n = 2, r = \frac{1}{a^3}$, therefore the Area
" ABD $= \frac{y^3v^2}{3a^3} - \frac{5}{6}v^2 + v\sqrt{2ay - y^2} \times \overline{\frac{2y^2}{9aa} + \frac{5y}{9a} + \frac{5}{3}} - \frac{2y^3}{27a} - \frac{5y^2}{18}$
" $- \frac{5ay}{3}$.

Dr. *Cheyne* has given us one general Theorem including all theſe, and of this Sort ad Inf. which we ſhall next produce.

Suppoſe now this Fluxion, *viz.* (the ſame with Dr. *Cheyne*'s) $\dot{y}\times ry^{n}\times F:\ \overline{\frac{ay}{\sqrt{2ay+y^{2}}}}\Big|^{m}$ or $ry^{n}v^{m}\dot{y}$ given: From the firſt aſſum'd Series (after Mr. *Craig*'s Way) adfected with indetermin'd Coefficients, (having taken the Fluxions) there will remain a ſecond Series of Fluxions, the Terms whereof involve v^{m-2}, for which you muſt aſſume a ſecond Series or Fluent, whence again (having taken the Fluxions) there will remain, after Compariſon, a third Series of Fluxions involving v^{m-3}, and ſo ad Inf. So that the Fluent will be

1. $$\frac{rv^{m}y^{n+1}}{n+1}-bv^{m}+\sqrt{\overline{2ay-yy}}\times\frac{mra}{\overline{n+1}|^{2}}y^{n}v^{m-1}$$

$$+\frac{Aa\times\overline{2n+1}}{n}y^{n-1}v^{m-1},$$ &c. ad Inf.

2. $$\frac{m\times\overline{m-1}\times ra^{2}}{\overline{n+1}|^{3}}y^{n+1}v^{m-2}-\frac{\overline{m-1}\,Aa^{2}\times\overline{2n+1}}{n^{2}}y^{n}v^{m-2},$$ &c.

ad Inf. &c. $+\ cv^{m-2}$.

3. $$\frac{-m\times\overline{m-1}\times\overline{m-2}\times ra^{3}}{\overline{n+1}|}y^{n}v^{m-3}-$$

$$\frac{\overline{m-1}\times\overline{m-2}\,Aa^{3}-n^{2}Ea\times 2n+1}{n^{3}}y^{n-1}v^{m-3}\times\sqrt{\overline{2ay-yy}},$$ &c.

4. $$\frac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\,ra^{4}}{\overline{n+1}|^{5}}y^{n}v^{m-4}$$

$$+\frac{\overline{\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times Aa^{4}+\overline{m-3}\times n^{2}Ea^{2}\times 2n+1}}{n^{4}}y^{n-1}v^{m-4}$$

$+$ &c. $-\ dv^{m-4}$.

5. $$\frac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\overline{m-4}\,ra^{5}}{\overline{n+1}|^{6}}y^{n}v^{m-5},$$ &c.

$\times\sqrt{\overline{2ay-yy}}$, &c. &c. ad Inf.

Upon this Theorem Dr. Cheyne *gives the following Obſervations.*

1. "THAT it conſiſts of an infinite Number of infinite Series, the firſt whereof has all the Terms beſides the ſecond adfected with the Sign +, the two following have all their Terms adfected with the Sign —; then the two next following with the Sign +, and ſo on changing alternately from two to two, except one Term always in the Even Series.

2. "That in each of the Terms now mentioned of the Even Series, and in the firſt Series, *viz.* in the ſecond Term of the firſt Series bv^m, mb is = equal to the laſt abrumpent Term of that Series; and in cv^{m-2} of the ſecond Series $m-1c$ is = to the laſt abrumpent Term of the third Series; in dv^{m-1} of the fourth Series $\overline{m-2}d$ is = to the laſt abrumpent Term of the fifth Series, and ſo in the reſt.

3. "That the particular Series are alternatively abſolute, and multiply'd by $\sqrt{2ay-yy}$, *viz.* the firſt Series (except the two firſt Terms) is multiply'd by $\sqrt{2ay-yy}$; the ſecond is abſolute; the third is multiply'd by $\sqrt{2ay-yy}$; the fourth is abſolute, and ſo on.

4. "That the great Letters every where obtain the ſame Value, and that wherever there comes up a new one in any Series, it is always the whole precedent Coefficient of that Series, and conſequently it will be eaſy to continue the Series as far as you pleaſe only by bare Obſervation, *v. g.* the ſecond Term of the ſeventh Series will be

$$\frac{-\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\overline{m-4}\times\overline{m-5}\times\overline{m-6}\times Aa^7-m-3\times m-4\times m-5\times m-6\times n^2\times Ea^5-\overline{m-5}\times\overline{m-6}\times n^4\times Ia^3-n^6\times Na\times 2n+1}{n^6}\,y^n v^{m-7}.$$

5. "That the Number of Series becomes finite when *m* is equal to an Integer and Affirmative Number, or to Nothing.

6. "That the Number of Terms in the particular Series will become finite, when *n* is equal to an Integer Affirmative Number, or to Nothing; or when $2n$ is equal to an Affirmative odd Number, or to Nothing.

7. "That except both *m* and *n* are equal to ſome Integer and Affirmative Number or to Nothing, or $2n$ to an Affirmative odd Number or to Nothing, the Fluent will not be exhibited in a finite Number of Terms.

Suppoſe for an Exemple the precedent Theorem of Mr. *Craig*, viz. that for rvy^n, and 'twill be found a particular Caſe of this. For you'll have $m=1$, and $n=n$, and $\frac{rv^my^{n+1}}{n+1}=\frac{rvy^{n+1}}{n+1}.-bv^m=-qv.\sqrt{2ay-yy}\times\frac{mra}{\overline{n+1}|^2}=\sqrt{2ay-yy}\times\frac{ra}{\overline{n+1}|^2y^n}$ (for here v by reaſon of $m-1=0$ vaniſhes) and ſo on.

Dr. Cheyne's *Example in Rud.* p. 22.

Thus $v^3y\dot{y}$, making $m=3$, $n=1$, $r=1$, gives $\frac{1}{2}v^3y^2-bv^3+\sqrt{2ay-yy}\times\frac{3}{4}av^2y+\frac{9}{4}a^2v^2$ (here the Series breaks off by reaſon of $n-1=0$ in the following Terms) $\frac{3}{4}a^2y^2v-\frac{9}{2}a^3yv$ (here alſo the ſecond Series ſtops) $+Cv-\frac{3}{8}a^3y-\frac{27}{8}a^4$. And here the Fluent ends, and becauſe $\frac{9}{4}a^2y^2$ is the laſt abrumpent Term in the firſt Series, therefore $3b=\frac{4}{9}a^2$, and conſequently $b=\frac{3}{4}a^2$. Likewiſe $\frac{27}{8}a^4$ is the laſt abrumpent Term in the third Series, therefore $c=\frac{27}{8}a^4$; wherefore the true Fluent is $\frac{1}{2}v^3y^2-\frac{3}{4}a^2v^3+\sqrt{2ay-yy}\times\overline{\frac{3}{4}av^2y+\frac{9}{4}a^2v^2}$ $-\frac{3}{4}a^2y^2v-\frac{9}{3}a^3y^2+\frac{27}{8}a^4v-\overline{\frac{3}{8}a^3y-\frac{27}{8}a^4}\times\sqrt{2ay-yy}$. Again, $v^3y^{\frac{1}{2}}\dot{y}$, wherein $m=3$, $n=\frac{1}{2}$, $r=1$, rightly apply'd to the Theorem will give $\frac{2}{3}v^3y^{\frac{1}{2}}-bv^3+\sqrt{2ay-yy}\times\overline{\frac{4}{3}ay^{\frac{1}{2}}v^2+\frac{16}{3}a^2y^{-\frac{1}{2}}v^2}-\frac{16}{9}a^2y^{\frac{3}{2}}v$ $-\frac{64}{3}y^{\frac{1}{2}}v+Cv,-\frac{33}{27}a^3y^{\frac{1}{2}}-\frac{1024}{27}a^4y^{-\frac{1}{2}}$ where becauſe $\frac{16}{3}a^2$ is the Coefficient of the laſt abrumpent Term in the firſt Series $3b=\frac{16}{3}a^2$, and conſequently $b=\frac{16}{9}a^2$, and alſo $c=\frac{1024}{27}a^4$. Thus $z=ry^{-n}v$ where v denotes the Arch of a Circle whoſe Radius is a, and its Ordinate $\sqrt{2ay-yy}$) will give the Area

$$=\frac{ry^{-n+1}v}{-n+1}+\sqrt{2ay-yy}\times\frac{-r}{3-2n\times1-n}y^{-n+1}+\frac{\overline{n-2}\times-A}{5-2\times a}y^{-n+2}$$

$$+\frac{\overline{n-3}\times-B}{\overline{7-2n}\times a}y^{-n+3}+\frac{\overline{n-4}\times-C}{9-2n\times a}y^{-n+4}+\frac{\overline{n-5}\times-D}{11-2n\times a}y^{-n+5}$$

$$\frac{n-6+-E}{\overline{13-2n}\times a}y^{-n+6},\ \&c.$$

Suppoſe now the Fluent of the given Fluxion $v^3\dot{x}$ is to be inveſtigated independently on the Theorem. Make Av^3x = to the firſt Term, φ: v will be $= \frac{r\dot{x}}{\sqrt{2rx - x^2}}$. And φ: Av^3x will $= Av^3\dot{x} + 3Av^2\dot{v}x$. $Av^3\dot{x}$ for $= v^3\dot{x}$. $A = 1$, there remains $3A^2\dot{v}x = \frac{3Av^2rxx}{\sqrt{2rx-xx}}$ for which aſſume $Bv^2\sqrt{2rx - xx}$. And its Fluxion will be $= -\frac{Bv^2x\dot{x}}{\sqrt{2rx-xx}}$ (to be compar'd with $\frac{3Av^2rx\dot{x}}{\sqrt{2rx-xx}}$) $+ \frac{Brv^2\dot{x}}{\sqrt{2rx-xx}}$, and the third Term of the Fluent comes out (from the ſecond) Cv^3, whereof the Fluxion is $3Cv^2\dot{v} = \frac{3Cv^2r\dot{x}}{\sqrt{2rx-xx}}$ which correſponds to $\frac{Brv^2\dot{x}}{\sqrt{2rx-xx}}$ for the remaining Term $\frac{Brv\dot{x}\sqrt{2rx-xx}}{\sqrt{2rx-xx}} = Brv\dot{x}$ aſſume Dux the firſt Term of whoſe Fluxion is $= \frac{Dv\dot{x} \times \sqrt{2rx-xx}}{\sqrt{2rx-xx}} = Dv\dot{x}$; and the ſecond Term $= \frac{Drx\dot{x}}{\sqrt{2rx-xx}}$ gives the next Term of the Fluent $= E\sqrt{2rx-xx}$, the firſt Term of whoſe Fluxion $- \frac{Ex\dot{x}}{\sqrt{2rx-xx}} = \frac{Drx\dot{x}}{\sqrt{2rx-xx}}$. The Second $\frac{Er\dot{x}}{\sqrt{2rx-xx}}$ gives Fv for the next Term, &c. For its Fluxion $\frac{Fr\dot{x}}{\sqrt{2rx-xx}} = \frac{Er\dot{x}}{\sqrt{2rx-xx}}$: And thus (having finiſhed the Compariſons) you'll have

$v^3x + 3v^2 \times \sqrt{2rx - xx} - rv^3 - 3r^2vx - 3r^3 \times \sqrt{2rx-xx} + 3r^3v$ = to the Fluent.

CHAP.

* * *

CHAP. IX.

Of Fluxional Æquations.

NOW we come to *Fluxional Æquations*, wherein alſo Sir *Iſaac Newton* was long before any other, as hinted in the firſt Chapter: He lays down two Methods for this Sort of Æquations, the one more particular, and as he ſays more conciſe, the other more general: We ſhall begin from the latter, which Sir *Iſaac* in the latter of his Letters to Mr. *Oldenburg* ſays, *conſiſts in aſſuming a Series for every known Quantity, whence all the reſt may be derived; and in comparing the Homologous Terms of the reſulting Æquation in order to find the Terms of the aſſum'd Series.* This Method Mr. *Leibnitz* alſo pretended to find out a long time afterwards, *viz*, in the Year 1684. We ſhall here take the Example made uſe of by *Leibnitz*, which is alſo Sir *Iſaac Newton*'s for the other Method. But here you are firſt to note, whether in the propos'd Fluxion there be any known (or determin'd) Quantity, for in that Caſe the Series to be aſſum'd muſt begin with an invariable or conſtant Quantity; if otherwiſe, with an indeterminate one adfected by an unknown Coefficient, and taking the Fluxions of theſe Aſſumptions, and æquating them to the Fluxions given, they will, after due Compariſon and Subſtitution, give the Fluents ſought. Sir *Iſaac Newton*'s Example, which Dr. *Cheyne* alſo makes uſe of, is this, Suppoſe an Æquation involving firſt Fluxions, *viz.* $y^2\dot{x}^2 - x^2x\dot{y} = dd\dot{x}^2 - dx\dot{x}^2$, wherein y is ſought by x; making $\dot{x} = 1$; and reducing the Æquation, you'll have $y^2 - x^2\dot{y} - dd + dx = 0$. Aſſume $y = a + bx + cx^2 + dx^3 + ex^4 + fx^5$, &c. then $y^2 = a^2 + 2abx + \overline{2ac + b^2}x^2 + \overline{2ad + 2bc}x^3$

$+\overline{2be+2af+c^2}\,x^4$, &c. And $-x^2\dot{y}=-bx^2-2cx^3-3ex^4$, &c. which being ſubſtituted in the Room of y^2 and $-x^2\dot{y}$, and ordering the Æquation, you'll have

$$o=\left\{\begin{array}{lccccc} y^2= & a^2+2abx & +2acx^2 & +2aex^3 & +2bex^4, & \&c. \\ & & +b^2 & +2be & +2af & \\ & & & & c^2 & \&c. \\ x^2\dot{y}= & & -b^2 & -2c & -3e^2 & \&c. \\ +dx= & +\ dx & & & & \\ -dd= & -\ dd. & & & & \end{array}\right\}=o$$

Now making each of the Terms equal to nothing, you'll have the Coefficients determin'd thus, $aa-dd=o$, & $a=d$; $2ab+d=o$, $b=-\frac{d}{2a}=-\frac{1}{2}$. $2ac+b^2-b=o$, and $c=-\frac{3}{8d}$, $e=-\frac{9}{16d^2}$, &c. wherefore ſubſtituting theſe determinate Values in the Room of the indeterminate ones in the aſſumed Æquation, you'll have $y=d-\frac{1}{2}x-\frac{3}{8d}x^2-\frac{9}{16d^2}x^3$, &c. But before we proceed any further, we will here lay down a Rule for Aſſumptions for all Fluxional Æquations ad Inf. 1. *Obſerve what we have already ſaid for aſſuming the firſt Term of the Series, and if all the Members of the Æquation are not of the ſame Order of Fluxions, make them ſo by ſubſtituting Fluxional Unities*; Then, 2. *Let the Number of Punctations, that is the Order of the Fluxion* (viz. *whether* 1*ſt*, 2*d or* 3*d*, &c.) *be made the Interval between the Indices of the indeterminate Quantity in the Aſſumption, and an Æquation ſo aſſum'd will always in its Fluxions compare with the given Fluxion*: Thus ſuppoſe $a^2\ddot{x}+x\dot{y}^2=o$. making $\dot{y}=1$, you'll have $a^2\ddot{x}+x=o$. Aſſume therefore $x=by+cy^3+dy^5+ey^7$, &c. If you had $a\dddot{x}=x=o$. Aſſume $x=by+cy^4+dy^7+cy^{10}$, &c. whence

$$o=\left\{\begin{array}{l} x=by+cy^4+dy^7+cy^{10}\ \&c. \\ a\dddot{x}=24cay+210day^4+720acy^7\ \&c. \end{array}\right\}=o$$

And aſſuming $b = 1$. you'll have $x = y + \frac{y^4}{24a} + \frac{y^7}{210a}$,&c. and ſo for $a\ddddot{x} + x = 0$, or $a\ddddot{x} + \dot{y}^5 x$ making $\dot{y} = 1$, you may aſſume $x = by + cy^6 + dy^{11}$, &c. &c. Suppoſe for another Example, That Æquation of *Leibnitz* for finding the Right Sine from the Arch and Radius given, his Æquation is $a^2\dot{y}^2 = a^2\dot{x}^2 + x^2\dot{y}^2$. Make $x = by + cy^3 + ey^5 + fy^7$,&c. and you'll have $\frac{\dot{x}}{\dot{y}} = b + 3cy^2 + 5ey^4 + 7fy^6$, &c. and comparing and ſubſtituting the due Values, you'll determine the Coefficients b, c, e, f, &c. But you'll do it much ſooner by deſcending to Second Fluxions. For making $\dot{y} = 1$, you'll have from the Æquation above, this, *viz.* $2a^2\dot{x}\ddot{x} + 2\dot{x}x\dot{y}^2 = 0$ or $a^2\ddot{x} + x\dot{y}^2 = 0$. and you'll have $\frac{\ddot{x}}{\dot{y}^2} = 2 \times 3cy + 4 \times 5ey^3 + 6 \times 7fy^5$, &c. as is evident by taking the ſecond Fluxions of the before-mention'd Value $\frac{\dot{x}}{\dot{y}}$, and the Æquation will be

$$0 = \left\{ \begin{array}{l} x = by + cy^3 + dy^5 + ey^7, \&c \\ \frac{a^2\ddot{x}}{\dot{y}^2} = 2 \times 3a^2cy + 4 \times 5dy^3 + 6 \times 7ey^5, \&c. \end{array} \right\} = 0$$

and determining the Coefficients, you'll have $x = \frac{y}{1} - \frac{y^3}{1 \times 2 \times 3a^2} + \frac{y^5}{1 \times 2 \times 3 \times 4 \times 5a^4} - \frac{y^7}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7a^6}$ &c. that is, making $a =$ Radius, $x =$ Right Sine, and $y =$ Arc, which for Practical Uſes (ſays he) ought to be remarkably leſs than Radius. Suppoſe now that Problem for raiſing any Series, or Multinomial Root to any indeterminate Power, or, which is all one, for extracting any Root of it: Suppoſe, for Example ſake, the Multinomial, or Infinitinomial Quantity $a + bx + cx^2 + dx^3$, &c, $= R$; & $\overline{a + bx + cx^2 + dx^3, \&c.}|^n$, $= R^n = y$, you'll have $b + 2cx + 3dx^2 + 4ex^3$, &c. $= \dot{R}$ & $nlR = ly$, & $\frac{n\dot{R}}{R} = \frac{\dot{y}}{y}$

and $ny\dot{R} - R\dot{y} = 0$, make

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4, \text{ \&c. you'll have}$$

$$\dot{y} = B + 2Cx + 3Dx^2 + 4Ex^3, \text{ \&c. and you'll have}$$

$$=0\left\{\begin{array}{l} ny\dot{R} = nbA + \left.\begin{array}{r} nbB \\ 2ncA \end{array}\right\}x \left.\begin{array}{r} +\ nbC \\ +\ 2ncB \\ +\ 3ndA \end{array}\right\}x^2 \left.\begin{array}{r} +\ nbD \\ +\ 2ncC \\ +\ 3ndD \\ +\ 4ncA \end{array}\right\}x^3 \text{ \&c.} \\ -R\dot{y} = -aB \left.\begin{array}{r} -\ bB \\ -\ 2aC \end{array}\right\}x \left.\begin{array}{r} -\ cB \\ -\ 2bC \\ -\ 3aD \end{array}\right\}x^2 \left.\begin{array}{r} -\ dB \\ -\ 2cC \\ -\ 3bD \\ -\ 4aE \end{array}\right\}x^3 \text{ \&c.} \end{array}\right\} = 0$$

Now for determining the Coefficients, make $A = a^n$, you'll have $nbA = aB$. $\frac{nbA}{a} = B = nba^{n-1}$; Et $nbB + 2ncA - bB - 2aC = 0$, &c. And $C = \frac{\overline{n-1} \times bB + 2ncA}{2a}$, &c. &c. which explicated into Terms, at length, will coincide with thoſe in the ſame Example in Mr. *Moivre* quoted at the beginning of this Treatiſe, as any one by Tryal will find.

By the ſame Method alſo you may transfer Series from one indefinite Quantity to another, as alſo perform Reverſion of Series (without conſidering Fluxions) &c. Suppoſe that Example of Sir *Iſaac Newton*, pag. 33: of his *Analyſ. per Quant. Inf.* &c. and writ heretofore in his ſecond Epiſtle to *Oldenburgh*, and publiſhed in the Third Volume of Dr. *Wallis*'s Works, *viz.* $z = ay + by^2 + cy^3 + dy^4 + ey^5$, &c. to find y in the Terms of z. Aſſume $y = Az + Bz^2 + Cz^3 + Dz^4$, &c. you'll have

$$0 = \left\{\begin{array}{lllll} ay = & aAz + & aBz^2 + & aCz^3 & \text{\&c.} \\ by^2 = & & bA^2z^2 + & 2bABz^3 + & \text{\&c.} \\ cy^3 = & & & cA^3z^3 + & \text{\&c.} \end{array}\right\} = 0$$

and making $a = 1$, you'll have $aA = 1$, and $A = \frac{1}{a}$. Again make $aB + bA^2 = 0$, or $aB = -bA^2$, and you'll have $B = \frac{bA^2}{a} = -\frac{b}{a^3}$. And making $aC + 2bAB + cA^3 = 0$, you'll have $C = -2bAB - cA^3 = \frac{2bb - ac}{a^5}$, and ſo on. And y will be $= \frac{z}{a} + \frac{b}{a^3}z^2 + \frac{2b^2 - ac}{a^5}z^3$, and ſo on. Again, make $z = ay + by^3 + cy^5 + dy^7$, &c. aſſume $y = Az + Bz^3 + Cz^5$, and you'll find $y = \frac{z}{a} - \frac{b}{a^4}z^3 + \frac{3b^2 - ac}{a^7}z^5 + \frac{8abc - a^2d - 12b^3}{a^{10}}z^7$, &c. And ſo if there were propos'd this Æquation expreſſing the Arch of a Circle, *viz.* $z = y + \frac{y^3}{6rr} + \frac{3y^5}{40r^4} + \frac{5y^7}{112r^6}$ + &c. You muſt aſſume $y = Az + Bz^3 + Cz^5 + Dz^7$, &c. and there will come out $y = z - \frac{z^3}{6r^2} + \frac{z^6}{120r^4} - \frac{z^7}{5040r^6}$ &c. and by this Method, ſays Sir *Iſaac Newton*, pag. 33. and after him *Leibnitz*, pag. 180. *Act. Erud.* the Roots of adfected Æquations may be extracted.

See Sir Iſaac, *Pag.* 34. *firſt Letter.*

Suppoſe now you have $az + bz^2 + cz^3$, &c. $= gy + by^2 + iy^3$, &c. to find z in the Terms of y: Make or aſſume $z = Ay + By^2 + Cy^3 + Dy^4 + Ey^5$, &c.

$$\begin{array}{llllllllllll}
az = & aAy & + aB & \Big\} y^2 & + aC & & + aD & & + aE & & + aF & \\
bz^2 = & \ldots & bA^2 & & + 2bAB & & + 2bAC & & + 2bAD & & + 2bEA & \\
 & & & & & \Big\} y^3 & bB^2 & & + 2bBC & & + 2bBD & \\
 & & & & & & & & & & bCC & \\
cz^3 = & \ldots & \ldots & & cA^3 & & + 3A^2B & \Big\} y^4 & + 3AB^2c & \Big\} y^5 & + 3A^2Dc & \\
 & & & & & & & & 3A^2Cc & & + 6ABCc & \Big\} y^6 \text{ \&c.} \\
 & & & & & & & & & & B^3c & \\
dz^4 = & \ldots & \ldots & & \ldots & & dA^4 & & + 2dA^3 & & + 2A^3Cd & \\
 & & & & & & & & & & A^2B^2d & \\
ez^5 = & \ldots & \ldots & & \ldots & & \ldots & & eA^5 & & + 5eA^4B & \\
fz^6 = & \ldots & \ldots & & \ldots & & \ldots & & \ldots & & fA^6 &
\end{array}$$

Now compare the Homologous Terms, and you'll have 1°. $aA = g$, or $A = \frac{g}{a}$. 2. $aB + bA^2 = b$, and $aB = b - bA^2$. Whence $B = \frac{b - bA^2}{a}$. 3. $aC + 2bAB + cA^3 = i$, whence $C = \frac{i - 2bAB - cA^3}{a}$. 4. $aD + 2bAC + bB^2 + 3cA^2B + dA^4 = k$.

Whence $D = \frac{k - 2bAC - bB^2 - 3cA^2B - dA^4}{a}$. Moreover you'll find $E = \frac{l - 2bAD - 2bBC - 3cAB^2 - 3cA^2C - 2dA^3 - eA^5}{a}$, &c. &c. Subſtitute theſe determinate Coefficients in the Room of the indeterminate ones in the aſſum'd Æquation, and you'll have the Fluent ſought.

Suppoſe this Logarithmical Fluxion given $\left\{\begin{array}{l} \dot{y}xlx + \dot{x}xy - \dot{x}lx = 0 \text{ ſeu Poſ. } \dot{x} = 1. \\ \dot{y}xlx + xy - lx = 0 \end{array}\right.$

Aſſume $y = alx + bxlx + cx^2lx + dx^3lx$. &c.

And you'll have $\dot{y} = ax^{-1} + blx + b + 2cxlx + cx + 3dx^2lx + dx^2$

$\dot{y}xlx = alx + bxl^2x + bxlx + 2cx^2l^2x + cx^2lx + 3dx^3lx$, &c.

That is $= alx + bxlx + cx^2lx +$ &c. &c.

And there will remain the firſt Fluxions of the higher Terms, *viz.* of

$bxl^2x\dot{x} + 2cx^2l^2x\dot{x}$, &c. For which you muſt aſſume $\alpha x^2l^2x + \beta x^3l^2x$, &c. &c. and thus the ſecond Aſſumption will ſpring from the firſt, &c.

This Method may be eaſily carry'd to higher Æquations, and ſuch as involve more unknown Quantities; but let theſe ſuffice.

The other Method of Sir *Iſaac Newton* conſiſts *in extracting the Fluent out of an Æquation that involves its Fluxion alſo:* Nor can this Method be propos'd more ſhort and conciſe than in his own Words. He propoſes this

PROBLEM.

To extract the Root out of an Æquation involving the Fluxion of the Root.

RESOLUTION.

“ MAke all the Terms, being placed on one Side of the Æquation, “ equal to Nought, and, if there be Occaſion, exalt or depreſs the Powers of y and $\dot{y}$, ſo that the Indices may be no where Negative, nor yet higher than ſhall be requiſite to this End; and let kz^λ

“ be the loweſt Term of thoſe that are not multiply'd by y nor $\dot y$, nor
“ by any Power of them. Let $lz^{\mu} y^{\alpha} \dot y^{\beta}$ be any other Term, and ſo running
“ over all the Terms in Order, collect out of each ſeparately the
“ Number $\frac{\lambda - \mu + \beta}{\alpha + \beta}$, ſo that you may have as many ſuch Numbers
“ as you have Terms. Call the greateſt of theſe Numbers ν, and z^{ν}
“ will be the Dignity or Power of the firſt Term of the Series. Make
“ its Coefficient a, and in the Æquation, which we call the Reſolvend,
“ write az^{ν} for y, and $\nu a z \nu - 1$ for $\dot y$; and having collected all the reſulting
“ Terms in which z is of the ſame Power as in the Term kz^{λ},
“ under their reſpective Signs, make them equal to Nought. And this
“ Æquation duely reduced will give you the Coefficient a. Thus you
“ have az^{ν} the firſt Term of the Series.

Second Operation.

“ For all the reſt of the Terms of this Series not yet found out put
“ p, and you'll have this Æquation $y = az^{\nu} + p$, and by the direct
“ Method $\dot y = \nu a \dot z z^{\nu - 1} + \dot p$. In the Reſolvend for y and $\dot y$ write theſe
“ their Values, and you'll have a new Reſolvend, wherein p ſtands in
“ the Room of y: And from this Reſolvend you may extract the firſt
“ Term of the Series p after the ſame way you extracted the firſt Term
“ of the whole Series $y = az^{\nu} + p$ out of the firſt Reſolvend.

The Third and following Operations.

“ Then you may find the third Reſolvend after the ſame way you
“ found the ſecond, and from that extract the third Term of the
“ whole Series.

“ And in like manner you'll find a fourth Reſolvend, and from that
“ a fourth Term of the Series, and ſo ad Infinitum. And the Series ſo
“ found will be the Root of the Æquation which was to be extracted.

EXAMPLE.

“ To extract the Root out of this Æquation (*which we have before*
“ *reſolved by the other Method*) viz. $y^2 \dot z^2 - z^2 \ddot z y - dd\ddot z z + d\dot z z^2 = 0$

“ Make $\dot{z} = 1$, and the Æquation will become $y^2 - z^2\dot{y} - dd + dz = 0$;
“ which is the Reſolvend. Now the loweſt Term in which neither y
“ nor $\dot{y}$ is found is dd, which made equal to kz^λ gives $\lambda = 0$. To the
“ other Terms $y^2, - z^2\dot{y}$ make $lz^\mu y^\alpha \dot{y}^\beta$ equal ſucceſſively, and thence in
“ the firſt place you'll have $\mu = 0$, $\alpha = 2$, $\beta = 0$, and in the ſecond
“ $\mu = 2$, $\alpha = 0$, and $\beta = 1$. And hence $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ will become in
“ the firſt Caſe 0, in the ſecond $= -1$; whence ν is 0, and az^ν and
“ $\nu az^{\nu-1}$ are a and 0; the two laſt whereof a & 0 being writ in the Re-
“ ſolvend for y and $\dot{y}$, produce $aa - 0z^2 - dd + dz$; and the Terms
“ aa and $-dd$, wherein the Index of the Power of z is λ or 0, be-
“ ing made $= 0$, give $a = d$. Whence the firſt Term of the Series
“ az^ν becomes d.

Second Operation.

“ For the remaining Terms put p, and you'll have this Æquation
“ $y = d + p$, and thence by the direct Method $\dot{y} = \dot{p}$, which Values
“ being ſubſtituted in the Reſolvend in the Room of y & $\dot{y}$, give the
“ new Reſolvend $2dp + pp - zz\dot{p} + dz = 0$, where p and $\dot{p}$ ſupply
“ the Room of y and $\dot{y}$. The only Term wherein is neither p nor $\dot{p}$ is
“ dz, which æquated to the Term kz^λ gives $\lambda = 1$. To the reſt of
“ the Terms $2dp$, pp, $- zz\dot{p}$ make $lz^\mu p^\alpha \dot{p}^\beta$ equal ſucceſſively; and
“ thence in the firſt Caſe you'll have $\mu = 0$, $\alpha = 1$, and $\beta = 0$; and
“ in the ſecond $\mu = 0$, $\alpha = 2$, $\beta = 0$; and in the third $\mu = 2$, $\alpha = 0$,
“ and $\beta = 1$. And hence $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ will become in the firſt Caſe 1,
“ in the ſecond $\frac{1}{2}$, in the third 0. Whence ν is 1, and az^ν, and $\nu az^{\nu-1}$
“ are az and a. The two laſt Terms az and a being written in the
“ Reſolvend, reſpectively produce $2daz + a^2z^2 - az^2 + dz$. And the
“ Terms $2daz$ and dz, wherein the Index of z is λ or 1, being made $= 0$
“ give $= -\frac{1}{2}$. Whence the firſt Term of the Series is $-\frac{1}{2}z$.

Third Operation.

“ For the reſt of the Terms not yet found put q, and you’ll have this “ Æquation $p=-\frac{1}{2}+q$, and thence by the direct Method $\dot{p}=-\frac{1}{2}+\dot{q}$: “ which Values, being ſubſtituted in the Room of p and $\dot{p}$ in the laſt “ Reſolvend produce a new Reſolvend $2dq-zq+qq-\frac{3}{4}zz-zz\dot{q}=0$, “ where q and $\dot{q}$ ſupply the Room of y and $\dot{y}$. The only Term in “ which neither q nor $\dot{q}$ is found is $\frac{3}{4}zz$, which compared with kz^{λ} gives “ $\lambda=2$; to the reſt of the Terms $2dq$, $-zq$, $+qq$, $-zz\dot{q}$ put $lz^{\mu}q^{\alpha}\dot{q}^{\beta}$ “ equal ſucceſſively; and thence in the firſt Caſe you’ll have $\mu=0$, “ $\alpha=1$, $\beta=0$; in the ſecond $\mu=1$, $\alpha=1$, $\beta=0$; in the third “ $\mu=0$, $\alpha=2$, $\beta=0$; in the fourth $\mu=2$, $\alpha=0$, $\beta=1$; and “ thence $\frac{\lambda-\mu+\beta}{\alpha+\beta}$ becomes in the firſt Caſe 2, in the ſecond, third, “ and fourth 1; and hence ν is 2, or az^{ν} and $\nu az^{\nu-1}$ are az^2 and $2az$, “ which Values being ſubſtituted in the Reſolvend for q and $\dot{q}$ give “ $2daz^2-az^3+aaz^4+\frac{3}{4}zz-2az^3$; and the Terms $2dazz+\frac{3}{4}zz$, “ wherein the Index of the Power of z is λ or 2, being made equal to “ Nought, give $a=-\frac{3}{8d}$. Whence az^{ν}, the firſt Term of the Series q, “ becomes $\frac{3zz}{8d}$.

Fourth Operation.

“ For the reſt of the Terms not yet found out put r, and you’ll have “ the Æquations $q=-\frac{3zz}{8d}+r$ & $\dot{q}=-\frac{3z}{4d}+\dot{r}$; and thence the “ new Reſolvend $2dr+\frac{9z^3}{8d}-zr+\frac{9z^4}{64dd}-\frac{zzr}{4d}+rr-zz\dot{r}=0$, and “ thence, by the foregoing Method, you’ll have $-\frac{9z^3}{16dd}$ the firſt Term of

" the Series r, and ſo ad Inf. Wherefore the Root to be extracted is
" $y = d + p = d - \frac{1}{2}z + q = d - \frac{1}{2}z - \frac{3zz}{8d} + r = d - \frac{1}{2}z - \frac{3zz}{8d}$
" $- \frac{9z^3}{16dd} -$ &c. and by continuing the Operation you may bring out
" the Root to more Terms.

Dr. *Cheyne* ſhews the Foundation of this Method, *pag.* 42, 43, 44. of his *Inverſe Method*, and how it may be extended to ſecond, third, and higher Fluxions, *pag.* 44, 45, 46; where he always puts (as Sir *Iſaac*) the loweſt Term $= kz^{\lambda}$, and for the reſt ſucceſſively in each Degree of Fluxions this Series

$$Lz^{\mu}y^{\alpha}\dot{y}^{\beta}\ddot{y}^{\gamma}$$

$$Lz^{\mu}y^{\alpha}\dot{y}^{\beta}\ddot{y}^{\gamma}\dddot{y}^{\delta}$$

$Lz^{\mu}y^{\alpha}\dot{y}^{\beta}\ddot{y}^{\gamma}\dddot{y}^{\delta}\ddddot{y}^{\epsilon}$ To which this Series anſwers that of the greateſt Numbers, *viz.*

$\frac{\lambda - \mu + \beta + 2\gamma}{\alpha + \beta + \gamma} = v =$ to the greateſt Number.

$\frac{\lambda - \mu + \beta + 2\gamma + 3\delta}{\alpha + \beta + \gamma + \delta} = v$

$\frac{\lambda - \mu + \beta + 2\gamma + 3\delta + 4\epsilon}{\alpha + \beta + \gamma + \delta + \epsilon} = v$, &c. ad Infinit. in which v anſwers the Power of z, to which being rais'd it will be = nearly to y.

You muſt put for $\ddddot{y}$ . . $v - 3 \times v - 2 \times v - 1 \times vz^{v-4}$
$\dddot{y}$ . . . . . . $v - 2 \times v - 1 \times vz^{v-3}$
$\ddot{y}$ . . . . $v - 1 \times vz^{v-2}$
$\dot{y}$ . . . . vz^{v-1}
y . . . . z^{v}

And in the following Page he accommodates it to an Æquation involving three indeterminate Quantities, *viz.* y, z, v with their Fluxions, *&c.*

Suppoſe now, for another Example of this Method, that of Dr. *Cheyne*, *pag.* 106. *viz.* this Æquation for the Area of an Hyperbola, *viz.* $A = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4}$, &c. or $-A + y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4}$, &c. $= 0$.
“ Now the loweſt Term in which there is neither y nor any Power of
“ it is A, which made equal to kA^{λ} gives $\lambda = 1$, and by making
“ $LA^{\mu}y^{\alpha}$ equal ſucceſſively to the reſt of the Terms $y, -\frac{y^2}{2}, +\frac{y^3}{3}, -\frac{y^4}{4}$, &c.
“ you'll have in the firſt Caſe $\mu = 0$, $\alpha = 1$, in the ſecond $\mu = 0$,
“ $\alpha = 1$, in the third $\mu = 3$, $\alpha = 3$, and ſo on. Hence $\frac{\lambda - \mu}{\alpha}$ is in the
“ firſt Caſe $= 1$, in the ſecond $= \frac{1}{2}$, in the third $= \frac{1}{3}$, &c. Where-
“ fore $\nu = 1$, $\alpha A^{\nu} = \alpha A$ which ſubſtituted in the Reſolvend in the
“ Room of y, gives $-A + \alpha A + \frac{\alpha^2 A^2}{2}$, &c. $= 0$, and conſequently
“ the Terms wherein A is of the ſame Power, being put $= 0$, give $\alpha = 1$.
“ whence the firſt Term of the Series is A. For the reſt put p, and you'll
“ have $z = A + p$, which Value being ſubſtituted in the Reſolvend in the
“ Room of y, gives $-A + A + p - \frac{A^2 + 2Ap + p^2}{2} +$
“ $\frac{A^3 + 3A^2p + 3Ap^2 + p^3}{3}$, &c. $= 0$. Now the loweſt Term in which
“ neither p, nor any Power of it is found is $-\frac{A^2}{2}$, which made
“ equal to kA^{λ} gives $\lambda = 2$, and by making $LA^{\mu}p^{\alpha}$ ſucceſſively equal
“ to the reſt of the Terms $p, -Ap, -\frac{p}{2}, +\frac{A^3}{3}$, &c. You'll have in
“ the firſt Caſe $\mu = 0$, $\alpha = 1$; in the ſecond $\mu = 1$, $\alpha = 1$; in the
“ third $\mu = 0$, $\alpha = 2$; in the fourth $\mu = 3$, $\alpha = 0$, &c. Whence $\frac{\lambda - \mu}{\alpha}$
“ in the firſt Caſe is 2; in the ſecond $\frac{1}{2}$; in the third 1, and ſo on.
“ Wherefore $\nu = 2$, $\alpha A^{\nu} = \alpha A^2$, ſubſtitute αA^2 in the Room of p, and
“ the Terms in which A is of the ſame Power being made equal to 0,
“ give $\alpha A^2 - \frac{A^2}{2} = 0$, and conſequently $\alpha = \frac{1}{2}$. For the reſt of the

“ Terms not yet found put q, and you'll have the Æquation $p = \frac{1}{2}\text{A}^2 + q$,
“ which ſubſtituted in the Reſolvend produces a new Reſolvend, which
“ treated as the others, will give the third Term of the Series $\frac{\text{A}^3}{6}$,
“ and by proceeding after this way according to the Precepts of the
“ Method above laid down, you'll have at length $y = \text{A} + \frac{\text{A}^2}{2} + \frac{\text{A}^3}{3}$
“ $\frac{\text{A}^4}{4}$, &c.

Having thus diſpatched theſe two Methods, there remains yet a third of *John Bernoulli*, which we will here give the Reader: It is printed in the *Act. Erudit.* of *Leipſick* in the Year 1694; Dr. *Cheyne* has it, pag 50 of his *Inverſe Method*. *The Fluent of a given Fluxion may be obtained by a general infinite Series, but yet for the moſt part in Terms very implicate.* He ſhews and inveſtigates the general Series it ſelf, but becauſe he gives a more particular Inveſtigation of it in his *Rudim.* we will only preſent the latter here, *viz. Suppoſe* $\dot{z}y$ *to be a general Expreſſion for all firſt Fluxions (where y denotes a Quantity any how compounded of indeterminate and conſtant ones) let its Fluent be* $\text{A}z^m y^n$ *then will* $m\text{A}z^{m-1}\dot{z}y^n + n\text{A}z^m y^{n-1}\dot{y}$ *be its Fluxion: Now the latter Term of the aſſumed Fluxion cannot be compared with the Fluxion given, becauſe there is no* $\dot{y}$ *in it, (and it is neceſſary in order to Compariſon that all the indeterminate Quantities, with their Fluxions, be found in the Fluxion of the aſſum'd Æquation that are in the propoſed one, and if there chance to be more, there will be no Danger, becauſe they being each made equal to* 1, *will vaniſh.) Wherefore the firſt Term of the Fluxion of the aſſumed Æquation muſt be compared with the Fluxion propoſed,* viz. $m\text{A}z^{m-1}\dot{z}y^n = \dot{z}y$, *whence* $m - 1 = 0$, $m\text{A} = 1$, $n = 1$. *And the firſt Term of the Fluent will be* zy, *the Fluxion whereof is* $\dot{z}y + z\dot{y}$, *but ſince there remains* yz, *with which no found Term of the Fluent can compare, I make it to be* $\text{B}z^m y^n$ *(not* y^n, *for ſo the Term to be found would coincide with that already found,) therefore* $m\text{B}\dot{z}z^{m-1}\dot{y}^n + n\text{B}z^m y^{n-1}\ddot{y}$ *is its Fluxion; but becauſe only the firſt Term can compare with the remaining Term of the Fluxion, from the firſt aſſumed Term (for otherwiſe* $\ddot{y}$ *would become equal to* 1, *and ſo the Compariſons would proceed no farther, for want of Homologous Terms, as will be found upon Trial)* *al) you'll have* $m - 1 = 1$, *and* $m = 2$, $n = 1$; $\dot z = 1 =$ *to a conſtant Quantity (wherefore it has no ſecond nor third,* &c. *Fluxions) and theſe being ſubſtituted in their Room, you have* $2Bz\dot y + \dot z y = 0$, *and* $B = -\frac{1}{2}$. *And ſo the ſecond Term of the Fluent becomes* $-\frac{z^2\dot y}{2}$; *but becauſe there remains here alſo the Term* $\frac{z^2\ddot y}{2}$, *with which there is none to compare, I aſſume* $Cz^m\ddot y^n$ *and ſo* $mCz^{m-1}\dot z\ddot y^n + nCz^m\ddot y^{n-1}\dddot y$ *will be its Fluxion. And becauſe by comparing the ſecond Term of this with the reſt, you have* $\ddot y = 1$, *and ſo the Compariſons would ſtop afterwards, therefore I compare the firſt with the Remainder, and I find* $m - 1 = 2$; *and conſeqnently* $m = 3$, $\dot z = 1$, $n = 1$, *and ſubſtituting theſe in their Room, you have* $3Cz^2\ddot y - \frac{1}{2}z^2\ddot y = 0$; *and determining the Coefficient, you have* $C = \frac{1}{1\times2\times3}$, *and ſo on. And conſequently the Fluent of* zy *is* $\dot z y - \frac{z^2\dot y}{1\times2} + \frac{z^3\ddot y}{1\times2\times3} - \frac{z^4\dddot y}{1\times2\times3\times4}$, &c. *and becauſe every where* $\dot z = 1$ *(that the Terms may become Homologous)* $zy - \frac{z^2\dot y}{1\times2\dot z} + \frac{z^3\ddot y}{1\times2\times3\dot z^2} - \frac{z^4\dddot y}{1\times2\times3\times4\dot z^3}$, &c. *and hence you may ſafely aſſume (having thus foreſeen from the Genius of the Fluxion the Fluent)* $azy + \frac{bz^2\dot y}{\dot z} + \frac{cz^2\ddot y}{\dot z^2}$, &c. *as far as you pleaſe. And* this Series (*ſays* Bernoulli) *will ſerve inſtead of a Canon to reſolve any Fluxions into a Series; for in particular Caſes* $\dot y$, $\ddot y$, $\dddot y$, $\ddddot y$, *&* $\dot z$, $\dot z^2$, $\dot z^3$, &c. *vaniſh, becauſe in any given Caſe their Relations are given in Algebraick Terms, and conſequently the Series in particular Caſes will conſiſt of Terms purely Algebraical. The Example he brings is this: Suppoſe* $\dot y = \frac{a\dot x}{r}$ *(where* $r = x + a$*) making in the Canon*

$z = x$, $\dot z = \dot x$, $y = \frac{a}{r}$, $\dot y =$ *(by reaſon of* $\dot r = \dot x$*)* $-\frac{a\dot x}{rr}$, $\ddot y = \frac{1\times 2ax^2}{r^3}$

$\dddot y = \frac{1\times 2\times a\dot x^3}{r^4}$, &c. *and you'll have* $y = \frac{ax}{r} + \frac{ax^2}{2r^2} + \frac{ax^3}{3r^3} + \frac{ax^4}{4r^4}$, &c.

Let $\dot z yv$ be a general Expreſſion for Tranſcendent or Exponential Quantities of the firſt Degree, (where y is an Algebraic Quantity compoſed any how of indeterminate and conſtant Quantities, and v any Quantity Geometrically irrational: Aſſume $\mathrm{A}z^m y^n v^p$ for the firſt Term, and you'll have from its Fluxion

$m\mathrm{A}\dot z z^{m-1}y^n v^p \,\big|\underline{+ n\mathrm{A}z^m\dot y y^{n-1}v^p + p\mathrm{A}z^m y^n \dot v v^{p-1}}$ for the firſt remaining Terms. (Make $\dot z yv = m\mathrm{A}\dot z z^{m-1}y^n v^p$, and you'll have $m - 1 = 0$, or $m = 1$, $n = 1$, $p = 1$, $m\mathrm{A} = 1$, $\mathrm{A} = \frac{1}{m} = 1$, and $\mathrm{A}z^m y^n v^p = zyv$; Now in the next place there remains $n\mathrm{A}z^m\dot y y^{n-1}v^p = n\mathrm{A}z^m\dot y v^p$ (by reaſon of $n - 1 = 0$.) $= z\dot y v$.

Aſſume $\mathrm{B}z^m\dot y^n v^p$ and you'll have

$m\mathrm{B}\dot z z^{m-1}\dot y^n v^p \,\big|\underline{+ n\mathrm{B}z^m\dot y^{n-1}\ddot y v^p \,\big|\underline{+ p\mathrm{B}\dot z^m y^n \dot v v^{p-1}}}$, & $m\mathrm{B}\dot z z^{m-1}\dot y^n v^p = z\dot y v$.

Whence $m - 1 = 1$, $m = 2$, $m\mathrm{B} = 1$, $\mathrm{B} = \frac{1}{2}$, $n = 1$, $p = 1$, and $\dot z = 1$.

Whence $\mathrm{B}z^m\dot y^n v^p = \frac{z^2\dot y v}{1\times 2}$; now there next remains $n\mathrm{B}z^m\dot y^{n-1}\ddot y v^p = n\mathrm{B}z^m\ddot y v^p$ $= \frac{z^2\ddot y v}{1\times 2}$. For which aſſume $\mathrm{C}z^m\ddot y^n v^p$, and you'll have $m\mathrm{C}z^{m-1}\ddot y^n v^p$

$\big|\underline{+ n\mathrm{C}z^m\ddot y^{n-1}\dddot y v^p \,\big|\underline{+ p\mathrm{C}z^m\ddot y^n \dot v v^{p-1}}}$ & $m\mathrm{C}z^{m-1}\ddot y^n v^p = \frac{z^2\ddot y v}{2}$, & $m - 1 = 2$,

$m = 3$, $m\mathrm{C} = \frac{1}{2}$, $\mathrm{C} = \frac{1}{2\times 3}$, $n = 1 = p$, whence $\mathrm{C}z^m\ddot y^n v^p = \frac{z^3\ddot y v}{1\times 2\times 3}$.

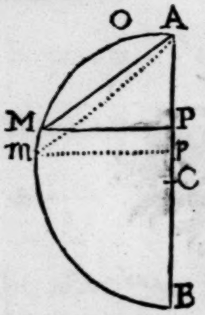
Now there remains again, $n\mathrm{C}z^m\ddot y^{n-1}\dddot y v^p = n\mathrm{C}z^m\dddot y v^p = \frac{z^3\dddot y v}{1\times 2\times 3}$. Aſſume $\mathrm{D}z^m\dddot y^n v^p$, and you'll have

$mDz^{m-1}\dddot{y}^n v^p + \Big|\underline{nDz^m\dddot{y}^{n-1}\ddddot{y}v^p \ \Big| \underline{+pDz^m y^n\dot{v}v^{p-1}}}$, and $mDz^{m-1}\dddot{y}^n v^p$ will be $=$ $\frac{z^3\dddot{y}v}{6}$. Whence $m-1=3$, and $m=4$, $n=1=p$, $mD=\frac{1}{6}$ & $D=\frac{1}{4\times 6}$, *&c.* after the ſame way will come out $Ez^m\ddddot{y}^n v^p = \frac{z^5\ddddot{y}v}{1\times 2\times 3\times 4\times 5}$ and $Fz^m\overset{\cdot\cdot\cdot\cdot\cdot}{y}^n v^p = \frac{z^6\overset{\cdot\cdot\cdot\cdot\cdot}{y}v}{1\times 2\times 3\times 4\times 5\times 6}$, *&c.* and there will remain this Series of Fluxions, *viz.* $pAz^m y^n\dot{v}v^{p-1} + pBz^m y^n\dot{v}v^{p-1} + pCz^m y^n\dot{v}v^{p-1} + pDz^m y^n\dot{v}v^{p-1}$, *&c. &c.* ad Inf. for which you muſt aſſume $Ez^m y^n v^p + Fz^m y^n v^p + Gz^m y^n v^p$, *&c.* ad Inf. So that this Fluent will be alſo Infinitely Infinite: And in all of them the firſt Aſſumption, whether (as it ſhall happen from the Genius of the Fluxion) a ſingle Quantity, as in ſome Caſes, or a whole Series as in others, gives or determines the reſt or ſubſequent ones.

CHAP. X.

Of the Method of finding Fluents *by* Geometrical Figures &c.

AS the Fluents of firſt Fluxions, handled as they ought to be, exhibit among other things the Area's of Curves; ſo alſo the Area's of Curves may repreſent or give the Fluents of firſt Fluxions: And hence you have a Method of finding Fluents by Curves, either Geometrical or Mechanical; ſo that to find the Fluents of firſt Fluxions is equivalent to the Quadrature of Curves, and reciprocally; or to ſquare Curves is the ſame thing as to find the Fluents of firſt Fluxions.



Suppoſe this Fluxion to be given; *viz.* $\dot{x}\sqrt{2rx - xx}$. From the Centre C, and Radius CB $= r$ deſcribe the Semicircle AMB, make AP $= x$, then will PB $= 2r - x$ and MP $= \sqrt{2rx - xx}$, and P*p* $= \dot{x}$, whence the Fluxion of the Area or the Parallelogram M*p* $= \dot{x}\sqrt{2rx - xx}$, and conſequently the Sum of all the $\dot{x}\sqrt{2rx - xx}$'s, or the Fluent is equal to the Semiſegment AMB.

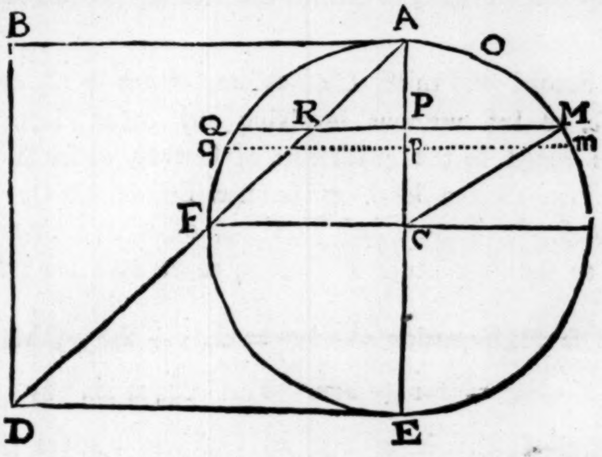
Suppoſe you are now to find the Fluent of $\frac{rxx}{2\sqrt{rx - xx}}$. Draw AM, and A*m* at an infinitely ſmall Diſtance from one another, and ſtanding at Right Angles on the Diameter AB, as alſo draw MR perpendicular to A*m*, then (by the Property of the Circle) AM $= 2rx$, and R*m* its

Fluxion is $= \frac{r\dot{x}}{\sqrt{2rx}}$. But becauſe the Triangles APM, MR*m* are ſimilar, (the Angles AMP and M*m*R ſtanding upon equal Arches) you'll have

$$PM\ (\sqrt{2rx-xx}) : AP\ (x) :: Rm\ \left(\frac{r\dot{x}}{\sqrt{2rx}}\right) : MR = \frac{rx\dot{x}}{\sqrt{2rx}+\sqrt{2rx-xx}}$$

and the infinitely ſmall Sector MAR $= \frac{1}{2}AR \times MR = \frac{rx\dot{x}}{2\sqrt{2rx-xx}} =$ to the given Fluxion, whence its evident that the Segment AOMA is the Fluent of the Fluxion given.

Suppoſe you are to find the Fluent of the Fluxion $x\dot{x} \times 2\sqrt{2rx-xx}$



From the Center C, and Radius CA $= r$ deſcribe a Circle AFEM, and make AP $= x$, P*p* $= \dot{x}$, PE $= 2r - x$ the Circumference of the Circle AFEM $= c$, and the Sum of all the $x\dot{x} \times 2\sqrt{2rx-xx} = \frac{crr}{2}$.

1. Suppoſe the firſt among Sir *Iſaac Newton*'s Examples, *viz.* an Æquation involving any Fluent and its firſt Fluxion without any other Fluent,

$$viz.\ aa\dot{v} = av + vv$$

and compleating the Dimenſions $av\dot{v} = av\dot{z} + vv\dot{z}$

Then $\frac{aav}{av + vv} = \dot{z}$, Make $\dot{v} = 1$,

and you'll have $\frac{aa}{av + vv} = \dot{z}$

And F : $\frac{aa}{av + vv} = z$ to the Area of the Curve, *i. e.* by ſquaring a Curve whoſe Ordinate is $\frac{aa}{av + vv}$ and Abſciſſe v, you'll have the Fluent z.

2. Thus for an Æquation involving two Fluxions of the ſame Fluent, without any other Fluent, as the firſt and ſecond, *viz.* $aa\ddot{v} = a\dot{v}\dot{z} + \dot{v}\dot{v}$: Or,

3. The Second and third, *&c.* as $aa\dddot{v} = a\ddot{v}\dot{z} + \dot{v}^3$, or third and fourth, *&c.* ad Inf. without involving any other Fluent; the Fluent may be found by the Quadrature of Curves, as ſee Sir *I. Newton's* Proceſs in *Prop.* 11. and *Schol.* at the latter End of his Quadratures:* And the Reaſon is, becauſe ſuch Fluxions, of what Degree ſoever, are equivalent to the Quadrature of Curves, as we have ſaid before.

4. Suppoſe this Æquation $a - bx^m = cxy^n\dot{y} + dy^{2n}\dot{y}\dot{y}$; Make $y^n\dot{y} = \dot{v}$, then will $a - bx^m = cx\dot{v} + d\dot{v}\dot{v}$. This Æquation by ſquaring a Curve, whoſe Abſciſſe is z, and its Ordinate $\dot{v}$ gives the Area v, and the other Æquation $\dot{v}, = y^n\dot{y}$, by taking its Fluent gives $v = \frac{1}{n+1}y^{n+1}$, whence you have the Fluent y.

5. Suppoſe $\overline{ax^m + bx^n}|^p = rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^t$ make $\dot{x} = 1$, and the latter part $rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^t$ by taking its Fluent be-

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* See Sir Iſaac's Scheme.

comes $ez^ry^s-\frac{f}{t+1}y^{t+1}$, which is as the Area of a Curve whoſe Abſciſſe is z, and its Ordinate $\overline{ax^m+bx^n}|^p$, and thence you have the Fluent y.

6. Suppoſe $\dot{x}\times\overline{ax^m+bx^n}|^p=\frac{d\dot{y}y^{n-1}}{\sqrt{ef+z^n}}$ and a Fluent whoſe Fluxion is $\dot{x}\times\overline{an^m+bx^n}|^p$ will be as the Area of a Curve whoſe Abſciſſe is x, and its Ordinate $\overline{ax^m+bx^n}|^p$, and a Fluent whoſe Fluxion is $\frac{d\dot{y}y^{n-1}}{\sqrt{e+fy^n}}$ will be as the Area of a Curve whoſe Abſciſſe is y, and its Ordinate $\frac{dy^{n-1}}{\sqrt{e+fy^n}}$.

CHAP.

CHAP. XI.

Of the Compariſon of Curves.

SIR *Iſaac* lays down in his 7th and 8th *Prop.* the Foundation of a Series of Propoſitions going on *ad Inf.* for the Compariſon of Curves. The 7th Propoſition runs thus.

Suppoſing the Series $e+fz^n+gz^{2n}+hz^{3n}$, &c. $=\mathrm{R}$, and in the Ordinate of any Curve $z^{\theta\pm\sigma n}\,\mathrm{R}^{\lambda\pm\tau}$, let there remain the given Quantities θ, n, λ, e, f, g, &c. and for σ and τ put ſucceſſively any Integer or whole Numbers: And if there be given the Area of one of the innumerable Curves that are denoted by the Ordinates ſo coming out, if the Ordinates are Binomials under the Vinculum of the Root, or if there be given the Areas of two of the Curves, if it be a Trinomial under the Root, the Areas of all the Curves in the Series will be given.

Suppoſe, for Example, the Ordinate of one [given] Curve to be $pz^{\theta-1}\,\mathrm{R}^{\lambda-1}$ and the Area $p\mathrm{A}$; R being a Binomial Quantity $e+fz^n$. And ſince [by *Prop.* 3. *viz.* in Sir *Iſaac*'s Book] $z^\theta\mathrm{R}^\lambda$ is the Area of a Curve whoſe Ordinate is $\left.\begin{array}{l}\theta e+\theta f\\ \qquad\lambda n\end{array}\right\}z^{\theta-1}\,\mathrm{R}^{\lambda-1}$ ſubtract the former Ordinate and Area from the latter, and there will remain

$$\begin{array}{l}\theta e+\overline{\theta+\lambda n}\,fz^n\\ -p\\ \hline \theta e-p+\overline{\theta+\lambda n}\,fz^n\times z^{\theta-1}\,\mathrm{R}^{\lambda-1}\end{array}\qquad\qquad\begin{array}{l}z^\theta\mathrm{R}^\lambda\\ -p\mathrm{A}\\ \hline z^\theta\mathrm{R}^\lambda-p\mathrm{A}=\end{array}$$

to the Ordinate and Area of a new Curve. Now make $\theta-p=0$, and the Ordinate will become $\theta+\lambda nfz^n\times\mathrm{R}^{\lambda-1}$ and the Area $z^\theta\mathrm{R}^\lambda-\theta e\mathrm{A}$; divide both by $\theta f+\lambda nf$, and you'll have $z^{\theta+n-1}$ for the Ordinate, and $\frac{z^\theta\mathrm{R}^\lambda-e\mathrm{A}}{\theta f+\lambda nf}=\mathrm{B}=$ to the Area; and aſſuming any Quantity q, $q\mathrm{B}$ will be the Area of a Curve $=\frac{qz^\theta\,\mathrm{R}^\lambda-q\theta e\mathrm{A}}{\theta f+\lambda nf}$ correſponding to the Ordinate $qz^{\theta+n-1}\mathrm{R}^{\lambda-1}$.

2. Sup-

2. Take now the Ordinate $\left.\begin{matrix}\theta f\\ \lambda n f\\ -q\end{matrix}\right\} z^{n} + \left.\begin{matrix}\theta g\\ 2\lambda n g\end{matrix}\right\} z^{2n} \times z^{\vartheta - 1} R^{\lambda - 1}$, and make $\theta f + \lambda n f - q = 0$; there remains therefore $\overline{\theta + 2\lambda n} g z^{2n} \times z^{\vartheta - 1} R^{\lambda - 1} =$ to the new Ordinate, and $z^{\vartheta} R^{\lambda} - \theta f B - \lambda n f B$ for the new Area. Divide each Part by $\theta g + 2\lambda n g$, and you'll have $z^{2n} \times z^{\vartheta - 1} R^{\lambda - 1} \dots \dfrac{z^{\theta} R^{\lambda} - \theta f B - \lambda n f B}{\theta g + 2n\lambda g}$

And aſſuming anyhow r you'll have $r z^{\theta + 2n - 1} R^{\lambda - 1} \dots \dfrac{r z^{\vartheta} R^{\lambda} - r\theta f B - r\lambda n f B}{\theta g + 2\lambda n g}$ $=$ to the new Ordinate and correſponding Area $= rC$.

3. Take now $\left.\begin{matrix}\theta g\\ 2\lambda n g\\ -r\end{matrix}\right\} z^{2n} + \left.\begin{matrix}\theta b\\ 3\lambda n b\end{matrix}\right\} z^{3n} \times z^{\vartheta - 1} R^{\lambda - 1} \dots\dots z^{\vartheta} R^{\lambda}$. You'll have $\theta g + 2\lambda n g - r = 0$, or $\theta g + 2\lambda n g = r$, and you'll find $\overline{\theta + 3\lambda n b} z^{3n} \times z^{\theta - 1} R^{\lambda - 1} =$ to the other new Ordinate and $z^{\vartheta} R^{\lambda} - rC =$ to its correſponding Area, that is (by reaſon of $r = \theta g + 2\lambda n g$) $z^{\vartheta} R^{\lambda} - \vartheta g C - 2\lambda n g C$. Divide now each Part by $\theta b + 3\lambda n b$, and you'll have $z^{3n} \times z^{\vartheta - 1} R^{\lambda - 1}$ and $\dfrac{z^{\theta} R^{\lambda} - \theta g C - 2\lambda n g C}{\theta b + 3\lambda n b} = D$, and aſſuming anyhow s you'll have $s z^{\vartheta + 3n - 1} R^{\lambda - 1}$, for the Ordinate, and for the Area $\dfrac{s z^{\vartheta} R^{\lambda} - s\theta g C - 2s\lambda n g C}{\theta b + 3\lambda n b} = sD$, and ſo on; ſo that a Series of Ordinates and Area's found after this way will exhibit it ſelf as follows.

Ord.	Area's.
$p z^{\theta - 1} R^{\lambda - 1} \dots\dots$	pA
$q z^{\theta + n - 1} R^{\lambda - 1} \dots\dots$	$z^{\theta} R^{\lambda} - pA = \dfrac{q z^{\theta} R^{\lambda} - q\theta e A}{\theta f + \lambda n f}$
$r z^{\theta + 2n - 1} R^{\lambda - 1} \dots\dots$	$z^{\theta} R^{\lambda} - qB = \dfrac{r z^{\theta} R^{\lambda} - \theta r f B - r\lambda n f B}{\theta g + 2\lambda n g}$
$s z^{\theta + 3n - 1} R^{\lambda - 1} \dots\dots$	$z^{\theta} R^{\lambda} - rC = \dfrac{s z^{\theta} R^{\lambda} - s\vartheta g C - 2s\theta n g C}{\theta b + 3\lambda n b}$, &c.

ad Inf. in the Binomial Kind.

Suppoſe now $R = e + fz^{n} + gz^{2n}$, for Trinomial Curves which are to be Squar'd by help of two given Curves, *viz.* from the given Ordinates $pz^{\theta-1}R^{\lambda-1}$ and $qz^{\theta+n-1}R^{\lambda-1}$, and the given Area's pA and qB.

$$\begin{array}{llr} & \text{Ord.} & \text{Ar.} \\ \text{Aſſume} & \theta e + \left.\begin{array}{c}\theta f \\ \lambda nf\end{array}\right\} z^{n} + \left.\begin{array}{c}\theta g \\ 2\lambda ng\end{array}\right\} z^{2n} \times z^{\theta-1} R^{\lambda-1} \;.\,. & z^{\theta} R^{\lambda} \\ \text{Subtract} & p \;.\,.\; q \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. & .\,.\; pA \;.\,.\; qB \end{array}$$

$$\overline{\left.\begin{array}{c}\theta e + \\ -p\end{array}\begin{array}{c}\theta f \\ \lambda nf \\ -q\end{array}\right\} z^{n} + \left.\begin{array}{c}\theta g \\ 2\lambda ng\end{array}\right\} z^{2n}} \times z^{\theta-1} R^{\lambda-1} \;.\,.\; \overline{z^{\theta} R^{\lambda} - pA - qB} =$$

to the new Ordinate and Area. Make now $\theta e - p = 0$, and $\theta f + \lambda nf - q = 0$, and the Ordinate will become $\overline{\theta g + 2\lambda ng}\, z^{\theta+2n-1} R^{\lambda-1}$, and the Area $z^{\theta}R^{\lambda} - \theta eA - \theta fB - \lambda nfB$. Divide each Side by $\theta g + 2\lambda ng$, and you'll have $z^{\theta+2n-1}R^{\lambda-1}$ . . . . $\dfrac{z^{\theta}R^{\lambda} - \theta eA - \theta fB - \lambda nfB}{\theta g + 2\lambda ng}$,

and aſſuming anyhow r, you'll have $rz^{\theta+2n-1}R^{\lambda-1}$ = to the Ordinate, and $\dfrac{rz^{\theta}R^{\lambda} - \theta reA - \theta rfB - \lambda nfB}{\theta g + 2\lambda ng} = rC$ = to the Area.

Suppoſe now the Ordinates $qz^{\theta+n-1}R^{\lambda-1}$ and $rz^{\theta+2n-1}R^{\lambda-1}$, and the Area's qB and rC given.

Aſſume $\left.\begin{array}{c}\theta e + \theta f \\ -p + \lambda nf \\ -q\end{array}\right\} z^{n} + \left.\begin{array}{c}\theta g \\ 2\lambda ng \\ -r\end{array}\right\} z^{2n} + \left.\begin{array}{c}\theta h \\ 3\lambda nh\end{array}\right\} z^{3n} \times z^{\theta-1}R^{\lambda-1}$, and the Area $z^{\theta}R^{\lambda}$.

Make $\theta e - p = 0$. $\theta f + \lambda nf - q = 0$, and $\theta g + 2\lambda ng - r = 0$. And you'll have $\overline{\theta h + 3\lambda nh}\, z^{3n} \times z^{\theta-1}R^{\lambda-1}$ = to the new Ordinate, and $z^{\theta}R^{\lambda} - qB - rC$ = to its Area, that is, $z^{\theta}R^{\lambda} - \theta fB - \lambda nfB - \theta gC - 2\lambda ngC$ = to the Area; and dividing each Part by $\theta h + 3\lambda nh$, you'll have

$z^{\theta+3n-1}R^{\lambda-1}$ and $\dfrac{z^{\theta}R^{\lambda} - \theta fB - \lambda nfB - \theta gC - 2\lambda ngC}{\theta h + 3\lambda nh}$, and aſſuming anyhow s you'll have $\dfrac{sz^{\theta}R^{\lambda} - \theta sfB - \lambda nsfB - \theta gsC - 2\lambda ngsC}{\theta h + 3\lambda nh}$ = to the Area of a Curve whoſe Ordinate is $sz^{\theta+3n-1}R^{\lambda-1}$.

Suppoſe now the Ordinates of 2 Curves $rz^{\theta+2n-1}R^{\lambda-1}$ & $sz^{\theta+3n-1}R^{\lambda-1}$, and their Area's rC and sD.

$$\text{Aſſume}\ \left.\begin{matrix}\theta e+\theta f\\ -p+\lambda nf\\ -q\end{matrix}\right\}z^{n}+\left.\begin{matrix}\theta g\\ 2\lambda ng\\ -r\end{matrix}\right\}z^{2n}+\left.\begin{matrix}\theta h\\ 3\lambda nh\\ -s\end{matrix}\right\}z^{3n}+\left.\begin{matrix}\theta k\\ 4\lambda nk\end{matrix}\right\}z^{4n}\times z^{\theta-1}R^{\lambda-1},$$

make now $\theta e-p=0$. $\theta f+\lambda nf-q=0$. $\theta g+2\lambda ng-r=0$. $\theta h+3\lambda nh-s=0$. and you'll have $\overline{\theta+4\lambda n}\times kz^{4n}\times z^{\theta-1}R^{\lambda-1}=$ to the Ordinate, and $z^{\theta}R^{\lambda}-r\text{C}-s\text{D}=$ to the Area. That is, $z^{\theta}R^{\lambda}-\theta g\text{C}-2\lambda ng\text{C}-\theta h\text{D}-3\lambda nh\text{D}=$ to the Area. Divide each Part by $\theta k+4\lambda nk$; and you'll have $z^{\theta+4n-1}R^{\lambda-1}$ and $\dfrac{z^{\theta}R^{\lambda}-\theta g\text{C}-2\lambda ng\text{C}-\theta h\text{D}-3\lambda nh\text{D}}{\theta k+4\lambda nk}$ and having anyhow aſſum'd t you'll have $tz^{\theta+4n-1}=$ Ordinate of a Curve whoſe Area is $\dfrac{tz^{\theta}R^{\lambda}-\theta tg\text{C}-2\lambda ntg\text{C}-\theta ht\text{D}-3\lambda nht\text{D}}{\theta k+4\lambda nk}$, and there is the ſame Reaſon and Way of proceeding for all Multinomials *ad Inf.* which fall under theſe Theorems. And any two being given (or taken any where) you may proceed either backwards or forwards as far as you pleaſe. But there remains the Second Caſe, which is this.

Caſ. 2. Suppoſe now the Ordinates $pz^{\theta-1}R^{\lambda}$. $qz^{\theta+n-1}R^{\lambda}$ multiply each by R, and divide it again by R, and you'll have

$\text{R})\,pz^{\theta-1}R^{\lambda+1}\,(pz^{\theta-1}R^{\lambda}=z^{\theta-1}R^{\lambda-1}\times pe+pfz^{n}+pgz^{2n}+phz^{3n}$, &c.

$\text{R})\,qz^{\theta-n+1}R^{\lambda+1}\,(qz^{\theta+n-1}R^{\lambda}=qz^{\theta-1}R^{\lambda-1}\times qe+qfz^{n}+qgz^{2n}$, &c.

Aſſume from the Third Prop. of Quadratures of Sir *Iſaac Newton* the intermediating Area's and Ordinates.

Ar. Ord.

$az^{\theta}R^{\lambda}$ . . . . $\overline{\theta ae+\theta af+\lambda naf}\,z^{n}+\left.\begin{matrix}\theta ag\\ 2\lambda nag\end{matrix}\right\}z^{2n}\times z^{\theta-1}R^{\lambda-1}$

$bz^{\theta+n}R^{\lambda}$ . . . . . . $\overline{\theta be+nbe}\,z^{n}+\left.\begin{matrix}\theta bf\\ nbf\\ \lambda nbf\end{matrix}\right\}z^{2n}\times z^{\theta-1}R^{\lambda-1}$, &c.

The Sum of theſe 4 Area's is $p\text{A}+q\text{B}+az^{\theta}R^{\lambda}+bz^{\theta+n}R^{\lambda}$ and the Sum of the correſponding Ordinates

θae

$$\left.\begin{matrix}\theta ae+\theta af\\ pe+\lambda naf\\ \theta be\\ pf\\ qe\end{matrix}\right\} z^{n} \qquad \left.\begin{matrix}\theta ag\\ 2\lambda nag\\ \theta bf\\ \lambda nbf\\ pg\\ qf\end{matrix}\right\} z^{2n} \qquad \left.\begin{matrix}\theta bg\\ nbg\\ 2\lambda nbg\\ qg\end{matrix}\right\} z^{3n}\times z^{\theta-1}\,R^{\lambda-1}$$

Make the firſt Term $= 0$ or $\theta ae + pe = 0$ and $\theta ae = -pe$ and $\theta a = -p$. Whence

2. $\theta af + \lambda naf + \theta be + nbe - \theta af + qe = 0$. Whence p vaniſhes in the Definition of q) Whence

3. $\theta ag + 2\lambda nag + \theta bf + nbf + \lambda nbf + pg + qf = 0$. Whence again p vaniſhes in the Deſinition of q.) Whence

4. $\theta bg + nbg + 2\lambda nbg + qg = 0$. And $qg = -\theta b - nb - 2\lambda nb$. Then by expunging (by the third) p and q

$\theta ag + 2\lambda nag + \theta bf + nbf + \lambda nbf - \theta ag - \theta bf - nbf - 2\lambda nbf = 0$, or $2\lambda nag - \lambda nbf = 0$, and $2\lambda nag = \lambda nbf$ or $2ag = bf$ and $\frac{2ag}{f} = b$. Whence the ſecond, viz. $\theta af + \lambda naf + \theta be + nbe + pf + qe$ becomes $\theta af + \lambda naf + \frac{2\theta age}{f} - \theta af - \frac{2\theta age}{f} - \frac{2nage}{f} - \frac{4\lambda nage}{f} = \frac{\lambda naff - 4\lambda nage}{f} =$ to the ſecond Term; and ſo the Sum of the four Ordinates is $\frac{\lambda naff - 4\lambda nage}{f} z^{\theta+n-1} R^{\lambda-1}$, and the Sum of as many correſponding Area's is $az^{\theta} R^{\lambda} + \frac{2ag}{f} z^{\theta+n} R^{\lambda} - \frac{\theta aA - 2\theta + 2n + 4\lambda n}{f} agB$ (for q is $= -\frac{2\theta ag}{f} - \frac{2nag}{f} - \frac{4\lambda nag}{f}$) Divide theſe Sums by $\frac{\lambda naff + 4\lambda nage}{f}$, that is, by the Coefficient of the Sum of the Ordinates, and if the latter Quotient be called D, D will be the Area of a Curve whoſe Ordinate is the former Quotient $z^{\theta+n-1} R^{\lambda-1}$, &c. Q. E. I.

Now for the Ordinate $z^{\theta-1} R^{\lambda-1}$

$$\left.\begin{matrix} -\theta b - nb - 2\lambda nb = q \\ \frac{\theta ag + 2\lambda nag + pg}{\lambda nf} = b \end{matrix}\right\}$$ then by exterminating the Quantities found

b and q

b & q in the 2d Term you find p, which is to be ſubſtituted in the 1ſt Term, viz. $\theta af + \lambda naf + \frac{\theta ag + 2\lambda nag + pg}{\lambda nf} \times \theta e + ne$ ($= b \times \overline{\theta e + ne} = \theta be + nbe$) $+ pf - \theta be - nbe - 2\lambda nbe = 0$ or $\theta af + \lambda naf + pf - 2\lambda nbe = 0$, and ſubſtituting b in $2\lambda nbe$ you'll have $\frac{\theta aff + \lambda naff - 2\theta age - \lambda nage - f}{2ge - ff} = p$ and $\frac{\lambda naffe - 4\lambda nagee - fe}{2ge - ff} = pe + \theta ae$. Whence making as before $-\theta b - nb - 2\lambda nb = q$ and $\frac{\theta ag + 2\lambda nag + pg}{\lambda nf} = b$, and $\frac{\theta aff + \lambda naff - 2\theta age - 4\lambda nage - f}{2ge - ff} = p$. you'll have, 1. $\frac{\lambda naffe - 4\lambda nagee - fe}{2ge - ff} z^{\theta - 1} R^{\lambda - 1} =$ to the Ordinate. And $az^{\theta} R^{\lambda} + \frac{\theta ag + 2\lambda nag + pg}{\lambda nf} z^{\theta + n} R^{\lambda} - \frac{\theta aff + \lambda naff - 2\theta age - 4\lambda nage - f}{2ge - ff} \times A$ ($-\overline{\theta b - nb - 2\lambda nb}\, B =$) $\frac{\theta ag + 2\lambda nag + pg}{\lambda nf} \times \overline{\theta - n - 2\lambda n} \times B$; And dividing on both Sides theſe Sums found, you'll have

$$\frac{az^{\theta} R^{\lambda} + \frac{\theta ag + 2\lambda nag + pg}{\lambda nf} z^{\theta + n} R^{\lambda} - \frac{\theta aff + \lambda naff - 2\theta age - 4\lambda nage - f}{2ge - ff} A - \frac{\theta ag + 2\lambda nag + pg}{\lambda nf} \times \frac{\theta - n - 2\lambda n}{\lambda nf} B}{\frac{\lambda naffe - 4\lambda nagee - fe}{2ge - ff}} =$$

C = to the Area, whoſe Ordinate is $z^{\theta - 1} R^{\lambda - 1}$.

SCHOLIUM.

You are to go on in a like Proceſs in the other Caſes of theſe Propoſitions, and you have a Way open'd to other Propoſitions, which go on in ſuch like Series ad Inf. As $pz^{\theta - 1} R^{\lambda}$ and $z^{\theta + n - 1} R^{\lambda}$ to which there anſwer'd pA and qB became $pe + pfz^{n} + pgz^{2n} \times z^{\theta - 1} R^{\lambda - 1}$ and $qez^{n} + qfz^{2n}$, &c. $\times z^{\theta - 1} R^{\lambda - 1}$, ſo $rz^{\theta - 1} R^{\lambda - 1}$ and $sz^{\theta + n - 1} R^{\lambda - 1}$, to which there anſwer rD and sC, if they are multiply'd as before by $e + fz^{n}$, and divided by the ſame, will become $\overset{(\alpha)}{r}z^{\theta-1}R^{\lambda-2} \times re + rfz^{n} + rgz^{2n}$, &c. and (β) $sqz^{\theta+n-1}R^{\lambda-2}\ sez^{n} + \times sfz^{2n}$, &c. ſo $\alpha z^{\theta-1}R^{\lambda-2}$, to which anſwer'd, &c. αE and βF if multipliy'd and divided as before, will become $\alpha z^{\theta-1}R^{\lambda-3}$ and $\beta z^{\theta+n-1}R^{\lambda-3}$, *&c. ad Inf.* and theſe Sorts of Proceſſes, as laid down in the 8th *Prop.* and others that may be form'd after the like manner, may be continu'd *ad Inf.*

Dr. *Cheyne*, inſtead of the Areas of Curves, compares the Fluents of given Fluxions together as follows. *See* Pag. 94.

Caſ. 1. Where the given Fluxions are expreſſed by the ſame Indeterminate Quantity, and included under no Vincula, or Radical Signs, you may make the Compariſon by common Diviſion, as the F: $x^{m}\dot{x}$ is to F: $x^{n}\dot{x}$ as $\frac{n+1}{m+1}x^{m-n}$.

Caſ. 2. Where the given Fluxions are expreſſed by the ſame Indeterminate, and are under the ſame Vincula, and have the ſame Quantities comprehended under thoſe Vincula, he makes one of the Fluxions $=\dot{A}$ to be compar'd with the other $\dot{B}$, and the Canon for the Fluent of $\dot{A}$ to be found by the before delivered Methods $=$ C, and the Coefficient of the laſt Term breaking off in that Canon $=$ Q, alſo the Canon for the Fluent of $\dot{B} =$ D and the Coefficient of the laſt abrumpent Term $=$ P; and, ſays he, you'll always have $A = mQB + C$ and $B = nPA + D$, and conſequently $A = \frac{mQB+C}{1-mnPQ}$, and $B = \frac{nPC+D}{1-mnPQ}$; alſo $\frac{A}{B} = \frac{mQD+C}{nPC+D}$; where m and n in any given Caſe will be given Quantities.

Caſ. 3. Where the given Fluxions are expreſs'd by the ſame Indeterminate, but either have different Vincula, or different Quantities comprehended under the Vinculum, or both different Vincula and different Quantities comprehended under them, you may compare by the precedent *Caſ.* both Fluents with the Fluent of the Fluxion ariſing from the Product of the Quantities denoted by the Vincula into the Fluxion of the Indeterminate Quantity, and conſequently you'll have the Relation of the one to the other. But here the Quantities of ſuch Fluents do not come out finite, according to the Conditions, except they degenerate into the firſt or ſecond Caſe.

CHAP.

CHAP. XII.

Of the Transformation of Curves.

THE next Principle of Sir *Iſaac Newton* (we ſhall here mention) for promoting the Quadrature of Curves is taken from a Propoſition in the Sixth Element of *Euclid*, *viz.* from the Conſideration that thoſe Rectangles are equal, whoſe Sides are reciprocally proportional, whence the Areas of thoſe Curves are equal, whoſe Ordinates are reciprocally as the Fluxions of the Abſciſſes: Becauſe the Elements of the Curves, or the Fluxions of the Area's, conſtitute ſuch Rectangles, when the Ordinates are parallel to one another, and at Right Angles with the Ax, and hence ariſes a Foundation for innumerable Compariſons, ſome whereof he explains in the following *Corollaries.* Suppoſe *e. g.* the Ordinate $z^{\theta-1} \times \overline{e+fz^{n}+gz^{2n}, \&c.}|^{\lambda}$. And taking any Quantity $=v$, and making $z^{\frac{n}{v}} = x$, z^{n} will $= x^{v}$, whence $n\dot{z}z^{n-1} = v\dot{x}x^{v-1}$ and $\dot{z} = \frac{v\dot{x}x^{v-1}}{nz^{n-1}}$ and (by reaſon of $x^{v-1} = z^{n-1}$) you have alſo $\dot{z} = \frac{v\dot{x}}{n}$. But $\dot{x} : \dot{z} :: z^{\theta-1} : \frac{\dot{z}z^{\theta-1}}{\dot{x}}$ & $z^{\theta-1} = x^{\frac{v\theta-n}{n}}$. Whence $\dot{z}z^{\theta-1} = \frac{\frac{v\dot{x}}{n}x^{\frac{v\theta-n}{n}}}{\dot{x}} = \frac{v}{n}z^{\frac{v\theta-n}{n}}$, &c. & *Coroll.* 3 & 4. follow after the ſame Method. Suppoſe in *Coroll.* 5. $z^{\theta-1} \times \overline{e+fz^{n}+gz^{2n}}|^{\lambda}$

make $\frac{1}{z} = x = z^{-1}$

then will $x^{\theta} = z^{-\theta}$

or $x^{-\theta} = z^{\theta}$

and $x^{-\theta-1} = z^{\theta-1} = x^{\frac{1}{\theta+1}}$

After the ſame Way again, becauſe

$$\frac{1}{z}=x=z^{-1}$$

$$\text{or } \frac{1}{z}=x \text{ and } 1=zx$$

$$\text{and } \frac{1}{x}=x^{-1}=z$$

$$\text{and } x^{-n}=z^{n} \text{ whence}$$

$$z^{\theta-1}\times\overline{e+fz^{n}+gz^{2n}}|^{\lambda}=x^{\frac{1}{\theta+1}}\times\overline{e+fx^{-n}+gx^{-2n}}|^{\lambda}=x^{\frac{1}{\theta+1+2\lambda n}}\times$$

$\overline{g+fx^{n}+ex^{2n}}|^{\lambda}$. After the ſame way $z^{\theta-1}\times\overline{e+fz^{n}+gz^{2n}+bz^{3n}}|^{\lambda}=x^{\frac{1}{\theta+1}}$ $\times\overline{e+fx^{-n}+gx^{-2n}+bx^{-3n}}|^{\lambda}=x^{\frac{1}{\theta+1+3\lambda n}}\times\overline{e+fx^{n}+gx^{2n}+bz^{3n}}|^{\lambda}$ &c. & ſo for the reſt. So that now for any given Fluxion, or Element of a Curve, ſuppoſe it was required to transform it into any other, which ſhall be the Element or Fluxion of an Area of another Curve equal to the former.

Aſſume a new indeterminate Quantity x in any given Reaſon to the former z (which is ſuppos'd to be the Ax in the given Curve to be transform'd ;) and this aſſum'd x will be the Axis in the Curve ſought. Now,

1. Find the Values of z and y expreſſed in the Terms of x.
2. Take the Fluxion of the new Abſciſſe x.
3. Make this Proportion, *viz*, Let the firſt Term be the Fluxion of the new Abſciſſe ; let the Fluxions of the firſt Abſciſſe and Ordinate expreſs'd by the new z be the ſecond and third Terms, and the fourth Term that ariſes from hence will be the Ordinate of the Curve chang'd, and the Rectangle of the Extremes will be the Element of the tranſmutated Area, and the Rectangle of the Means will be the Element of the Curve propos'd.

Suppoſe now the 2d Ordinate in the firſt Table of Sir *Iſaac Newton*, viz. $\frac{dz^{n-1}}{e+fz^{n}}$, now to transform this Curve into another equal to it, and whoſe Axis is x. Aſſume $x=z^{n}$. Thence $z=x^{\frac{1}{n}}$, and $z^{n-1}=x^{1-\frac{1}{n}}$ $\therefore$ $\dot{x}=\frac{1}{n}\dot{z}z^{n-1}$ & $\frac{dz^{n-1}}{e+fz^{n}}=\frac{dx^{1-\frac{1}{n}}}{e+fx}$. Now make this Proportion $\dot{x}:\dot{z}\ (=\frac{1}{n}x^{\frac{1}{n}-1}\dot{x})::\frac{dz^{n-1}}{e+fz^{n}}\ (=\frac{dx^{1-\frac{1}{n}}}{e+fx}):\ \frac{d}{n\times\overline{e+fx^{n}}}$ the fourth Term will here be the Ordinate of the Curve tranſmutated or chang'd; ſince the Element, *viz.* the Rectangle of the Extremes is equal to the Rectangle of the Means; and the Rectangle of the Means is the Element of the Curve propos'd.

Therefore the F : $\frac{d\dot{x}}{n \times \overline{e+fx}}$ or the Area of the transform'd is equal to the Area of the propos'd, *viz.* to the Area of the Curve whoſe Ordinate is $\frac{dz^{n-1}}{e+fz^n}$. This may be ſometimes perform'd more conciſely by ſetting down for the firſt Term of the Proportion the Fluxion $\dot{x}$ expreſs'd by z, for the ſecond and third $\dot{z}$ and $\frac{dz^{n-1}}{e+fz^n}$ and the fourth that comes out thence will be the Ordinate of the transform'd, afterwards expreſs'd by x. In the laſt Example where $z^n = x$ the Proportion will be $\dot{x} = n\dot{z}z^{n-1} : \dot{z} :: \frac{dz^{n-1}}{e+fz^n} : \frac{d}{n \times \overline{e+fz^n}}$ and ſubſtituting x in the Room of z^n the Ordinate of the transform'd will be $\frac{d}{n \times \overline{e+fx}}$. And the Element of the ſaid Area will be $\frac{d\dot{x}}{n \times \overline{e+fx^n}}$. But $\frac{d\dot{x}}{n \times \overline{e+fx}} = \frac{1}{n} \times \frac{d\dot{x}}{e+fx}$ (and ſuppoſing d to be the Square in the Æquilateral Triangle, *i. e.* that $d =$ * AG x KG, and that $ef = 1$, $e =$ KF, $x =$ FI) is the Element of the Quadrilateral Hyperbolical Space FI*if* multiply'd by $\frac{1}{n}$, and $\frac{1}{n} \times$ F : $\frac{dx}{e+fx}$ is that Quadrilateral Space ſuppos'd as Given. After this way the Fluent of the Fluxion $\frac{dz^{n-1}\dot{z}}{e+fz^n}$ is equal to the Hyperbolical Space between the Aſymtotes FI*if* of the Æquilateral Hyperbola multiply'd by $\frac{1}{n}$. And if the Reader has a mind, he may, from this Curve as given, find by the precedent Chapter the Fluents of the Fluxions $\frac{dz^{2n-1}\dot{z}}{e+fz^n}$, $\frac{dz^{3n-1}\dot{z}}{e+fz^n}$, &c. and when he has found them, regiſter them in a Table, as Sir *Iſaac Newton* has done.

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* *See Scheme at the latter End of this Treatiſe.*

And after the ſame way you may proceed for $d\dot{z}z^{\frac{1}{2}n-1}\times\overline{e+fz^{n}}|^{-r}$, &c. *viz.* by ſubſtituting $\sqrt{\frac{d}{e+fz^{n}}}=x$, &c. ad Inf.

Suppoſe $\frac{d}{z}\dot{z}\sqrt{e+fz^{n}}$ to be the Element of a Curve, whoſe Ordinate is $\frac{d}{z}\sqrt{e+fz^{n}}$; this depends on the Quadrature of the Hyperbola or the Ellipſe, or on the F: $-\dot{x}\sqrt{f+exx}$, by ſuppoſing $z^{\frac{1}{n}}$ or $z^{-n}=xx$. Whence $z^{-\frac{1}{2}n}=x$. $x^{\frac{1}{2}n}=x^{-1}$, & $z^{n}=x^{-2}$, & $-\frac{1}{2}nz^{-\frac{1}{2}-1}\dot{z}=\dot{x}$. Then $\dot{x}\,(-\frac{1}{2}nz^{-\frac{1}{2}n-1}\dot{x}:)\dot{z} : dz^{-1}\times\sqrt{e+fz^{n}} : \frac{2d}{-n}z^{\frac{1}{2}n}\sqrt{e+fz^{n}}$, and ſubſtituting x for z in the laſt Term, you'll have $\frac{2d}{-n}x^{-2}\times\sqrt{f+exx}$; and multiplying by x, you'll have for the Curve tranſmutated $\frac{2d}{-n}x^{-2}\dot{x}\times\sqrt{f+exx}$. And ſuppoſing as known F: $-\dot{x}\times\sqrt{f+ex^{2}}$; you'll find by the precedent Problem the Fluent of $\frac{2a}{-n}x^{-2}\dot{x}\sqrt{f+ex^{2}}$ *viz.* the Fluent of the Fluxion propos'd.

Let it now be propos'd to tranſmutate $\frac{d\dot{z}}{ff+zz}$ into this, *viz.* $-\frac{d\dot{x}}{2\sqrt{x-ffx^{2}}}$ (which is the Element of the Sector of a Circle where ff is taken for an Arbitrary Unity, which is alſo the Diameter of a Circle) by making $z=\sqrt{\frac{1}{x}-ff}$; whence $z^{2}=\frac{1}{x}-ff$ & $zz+ff=\frac{1}{x}$ & $\overline{zz+ff}|^{2}=\frac{1}{x^{2}}$. $\frac{1}{x}=\frac{1}{zz+ff}=\overline{zz+ff}|^{-1}$ & $xx=\overline{zz+ff}|^{-2}$ & $\dot{x}=-\frac{2z\dot{z}}{\overline{zz+ff}|^{2}}$.

$$\text{Then } \dot{x}\,\left(-\frac{2z\dot{z}}{\overline{zz+ff}|^{2}}\right) : \dot{z} :: \frac{d}{ff+zz} : \left.\frac{\frac{d\dot{z}}{ff+zz}}{\frac{-2z\dot{z}}{\overline{zz+ff}|^{2}}}\right\} = \frac{d\dot{z}\times\overline{zz+ff}|^{2}}{2ffz\dot{z}+2z^{3}\dot{z}}$$

=

$= \frac{d \times \overline{zz + ff}|^2}{2z \times ff + zz}$ (by ſubſtituting the Value of z) $- \frac{d}{2\sqrt{x - ffxx}}$ becauſe

$$\frac{-\frac{d}{x}}{2\sqrt{\frac{1}{x} - ff}} = - \frac{d}{2x\sqrt{\frac{1}{x} - ff}} = - \frac{d}{2\sqrt{xx}\sqrt{\frac{1}{x} - ff}} = - \frac{d}{2\sqrt{\frac{xx}{x} - ffxx}}$$

$= - \frac{d}{2\sqrt{x - ffxx}}$ and multiplying this Fourth Term by the firſt $\dot{x}$ you'll have $- \frac{d\dot{x}}{2\sqrt{x - ffxx}} =$ to the Element of the Sector of a Circle multiply'd by a conſtant Quantity $-a$. Which muſt yet be multiply'd by $\frac{1}{4}$, ſuppoſing f to be both an Arbitrary Unity, as well as Diameter of the Circle. In the laſt Place, Suppoſe the Element of that Curve (heretofore propos'd by *Leibnitz* to *Oldenburg*) $- \frac{8r^5xx\dot{x}}{\overline{rr + xx}|^3}$. Make $xx = \frac{2r^3 - rru}{u}$ then will $xxu = 2r^3 - rru$ and (by taking their Fluxions) $2x\dot{x}u + xx\dot{u} = -rr\dot{u}$ or $2x\dot{x}u = -rr\dot{u} - xx\dot{u}$ and $\dot{x} = - \frac{rr\dot{u} - xx\dot{u}}{2u\frac{r}{u}\sqrt{2ru - uu} = 2xu}$ or $\dot{x} = \frac{\overline{rr - xx} \times \dot{u}}{\frac{2ru}{u}\sqrt{2ru - uu}} = \frac{\overline{rr - xx}\dot{u}}{2r\sqrt{2ru - uu}}$.

$$\text{And } \dot{x} = \frac{\frac{rr\dot{u}}{1} - \frac{2r^3\dot{u} - rru\dot{u}}{u}}{2r\sqrt{2ru - uu}} = \frac{\frac{rru\dot{u} - 2r^3\dot{u} - rru\dot{u}}{u}}{2r\sqrt{2ru - uu}} = - \frac{2r^3\dot{u}}{2ru\sqrt{2ru - uu}}$$

$= - \frac{rr\dot{u}}{u\sqrt{2ru - uu}}$; and forming this Proportion $\dot{u} : \dot{x}$ $\left(= - \frac{ru\dot{u}}{u\sqrt{2r - uu}}\right)$ $:: - \frac{8r^5x^2}{\overline{rr + xx}|^3} \left(- \frac{2ruu + u^3}{rr}\right) : \frac{2ru - uu}{\sqrt{2ru - uu}} = \sqrt{2ru - uu}$, and you'll have $\dot{u}\sqrt{2ru - uu}$ for the Fluxion of the transform'd or chang'd Curve, and the F: $\dot{u}\sqrt{2ru - uu}$ will expreſs the Area of the Circle.

CHAP.

CHAP. XIII.

OR

The CONCLUSION.

A Synopſis or Review of the Inventions exhibited in the precedent Chapters, &c.

THAT we may now come to a Concluſion; It will appear evident from what has been ſaid, I ſuppoſe, that Sir *Iſaac Newton* was the firſt Author of this Method (under what Name ſoever it may appear;) and that *Leibnitz* did not fall upon it till long after, even upon Suppoſition that he had it not from his Correſpondence with Sir *Iſaac*, which many ſuſpect, and while theſe Papers have been under the Preſs, and almoſt brought to a Concluſion, there has been publiſh'd a Treatiſe, by Order of the *Royal Society*, containing the Commerce by Letters between Mr. *John Collins* and others concerning the Promotion of Analytical Affairs; wherein the whole Controverſy concerning the Author of *The Method of Fluxions*, or the *Differential Calculus*, is delivered in the Original Letters between Sir *Iſaac Newton*, Mr. *Leibnitz*, and others, and fully and clearly laid open. Wherefore in Room of what we had otherwiſe concluded with, we have choſe to repreſent the Matter as extracted from thence; whence it appears that Sir *Iſaac Newton* had this Method before the Years 1664 and 1665, and had exhibited the Extraction of Roots in Species, and alſo all Fluxions, as firſt, ſecond, third, *&c. ad Inf.* by this Series, *viz.* by $\overline{x+o}|^{\frac{m}{n}}$

$$x^{\frac{m}{n}} + \frac{m}{n} ox^{\frac{m-n}{n}} + \frac{m^2 - mn}{2n^2} oox^{\frac{m-2n}{n}} + \frac{m^3 - 3m^2n - 2mn^2}{6n^3} ooox^{\frac{m-3n}{n}}, \&c.$$

as may be ſeen in his firſt Epiſtle to *Oldenburgh* and his Schol. of Fluxions at the End of his Quadratures. He writ the firſt of theſe Epiſtles the 13th of *June*, 1676, and the other 23 *Octob.* the ſame Year, deſiring they might be communicated to Mr. *Leibnitz*. In the firſt, beſides the Extraction of Roots in Series, he gives the Quadrature of the

Quadratix, which plainly (to ſpeak in Mr. *Leibnitz*'s Stile) Involves the *Summatory* Method. In the *Second*, he Communicated the Invention of *Fluxions* and *Fluents* Conceal'd in *tranſpos'd* Letters (as was ſhewn at the beginning of this Treatiſe) as alſo two Methods for *Fluxional Equations*, with Examples of *Quadratures*, and an Intimation that he had for a good while, had Methods for comparing *Curves* with one another, *pag.* 76. 79. He modeſtly obſerves, that Mr. *Leibnitz*, in offering a Series for finding the *Coſine* from the *Arch*, could ſcarce take notice of the Series he had ſent but a little before, for finding the *Verſed ſine* from the ſame *Arch* given; becauſe theſe two are one and the ſame. He gives moreover (amongſt many others which he had for the ſame purpoſe) two *Theorems* for *Reverſion* from *Areas* to right Lines; but thought fit to Conceal *another Method* which he had for the ſame purpoſe. This was the *Direct* Method of *Fluxions*; and then he ſays, *that the Inverſe Method of Tangents, and other Problems more difficult then that, were alſo in his Power*, viz. By the help of thoſe *Fluxional* Methods, which he then decipher'd in an Ænigmatical ſort of way, to apt Interpreters, if there were any ſuch. But Mr. *Leibnitz* (as it there appears) had in the Year $167\frac{2}{3}$, another *Method of Differences*; and Mr. *Collins*, in or about the Year 1670, began to Communicate Mr. *Newton*'s, and in 1671, Mr. *Gregory*'s Series, to ſome Friends. It was in the Year 1673, that Mr. *Leibnitz* was in *London*, but he Communicated no Series at all there, but what he had receiv'd from Mr. *Oldenburgh*; to whom he writes from *Paris*, *March* 30. 1675. *That He underſtood from Him, Mr.* Newton *had the Invention of all Sorts of Quadratures, and the dimenſion of Surfaces and Solids generated by Revolution, and of Centers of Gravity, by certain Methods of Approximation, which Method* (ſays He) *if it be Univerſal and Commodious deſerves highly to be eſteem'd*; and he makes no doubt, *but 'tis every way worthy of the moſt Ingenious Author*. Nor is there any mention made at all of any diſcoveries of his own in *this Kind*. *May* 20 1675, He ſpeaks of ſome Series receiv'd from Mr. *Collins*, that they might be compar'd with *his own*. And the ſame Year 1675, He had a certain little Treatiſe compos'd about *Arithmetical Quadratures*, which, (ſays the Commentator upon the *Commercium Epiſtolicum*) was the very ſame, which Mr. *Gregory* Communicated to Mr. *Collins*, in the beginning of the Year 1671, and which Mr. *Oldenburgh* Communicated to Mr. *Leibnitz* this ſame Year. And he goes on (viz. the *Commentator* aforeſaid) ſpeaking of this Method of Quadrature, Mr. *Leibnitz has Compos'd a Treatiſe on this Subject after the Common manner, and began* this Year *to Communicate it to his Friends : The* next Year *He Poliſh'd and fitted it up, to Communicate to Mr.* Oldenburgh : *The* third Year, *returning back into his Country, He began to be concern'd in Publick Buſineſs, and the Work Swelling under his Hands too, He had not leiſure to beſtow any* *farther Preparations upon it, in order to its ſeeing the Light. Beſides, He thought it not worth while, to handle thoſe Things Prolixly in a* Vulgar *way, which his* New Analyſis, *would exhibit in a very little Compaſs.* Therefore, *This* New Analyſis, *was certainly diſcover'd*, after *Mr.* Leibnitz *had given over Poliſhing and Correcting his Work, Compos'd after a* Vulgar *Method, and* after *He began to be concern'd in Publick Affairs.* So that it was in the Year 1677, that Mr. *Leibnitz* firſt began to Communicate his *Differential Method* to his Friends; tho' He had had it (as He ſays) a great while by Him, and had perform'd the Buſineſs of *Tangents* by the *Differences of Ordinates*; But He had alſo perus'd thoſe two Letters of Mr. *Newton*, before-mention'd. And yet for all this, in his Letter *Aug.* 27. 1676. attempting the ſame Problem, *He delivers the Method of Solution in a Prolix and tedious way* (which as the *Commentator* well obſerves, He might have exhibited in very few Lines, by his *New Analyſis, provided He had then been Maſter of it*; From whence He concludes, that Mr. *Leibnitz* had not that Method at that Time) and asks farther, *How Mr.* Newton *went to Work in the Method of* Reverſions, *as when He finds the Number from the Logarithm*, &c. *wiſhes for a Demonſtration of Mr.* Gregory*'s Approximation*, &c. And ſays, *That various Problems are of ſo dark and perplex'd a Nature, that they neither depend upon* Equations, *nor* Quadratures; *of which Sort (amongſt many others) are thoſe of the* Inverſe Method of Tangents, *which* Cartes *own'd He was not Maſter of.* Mr. *Newton* reply'd, *that He could Solve all theſe Problems, by his Equations*; and upon this notice from Mr. *Newton*, then and not before, as the Remarker obſerves, Mr. *Leibnitz ſaw all this to be ſo. And* (He goes on) *if Mr.* Leibnitz, *had then known any Thing of Differential Equations, He had never ſaid, that the Problems of the* Inverſe Method of Tangents, *did not depend upon Equations.*

May 12. 1676. He receiv'd from Mr. *Collins*, by one *Mohr* a *Dane*, two Theorems of Mr. *Newton*'s, expreſſing the Relation of the *Arch* to its *Sine*, by Infinite Series; as follows, *viz.* putting the *Sine* $= x$, *Arch* $= z$, *Radius* $= 1$, then will

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9, \text{ \&c.}$$

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9, \text{ \&c.}$$

This He thought to be wonderfully Ingenious, and *deſired the Demonſtration of it*; which He afterwards had, at his requeſt, of Mr. *Newton*, together with the Series themſelves, and the Method of coming at them, and tho' He did not then ſo clearly underſtand them, *yet He found them afterwards amongſt his old* Papers. But He had been (it ſeems) accuſtomed to theſe Sort of Arts, before this:

For whereas Mr. *Gregory* (amongſt many other Series) diſcover'd one, for finding the Natural *Tangents* from the *Arks*, and *vice verſa*; viz. making *Radius* $= r$, the *Arch* $= a$, the *Tangent* $= t$, it will then be was $t = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2837r^8}$ &c. and converſely, from the Tangent to find its Arch $a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} -$ &c.

This Series (ſays the *Commentator*) *Mr.* Leibnitz *had from Mr.* Gregory, *in the beginning of the Year* 1671, *and paſs'd it amongſt his Friends in France*, as his own, *without taking any notice at all of Mr.* Oldenburg'*s Letter, in which He receiv'd it.*

More of this Sort of Correſpondence, and of theſe Artifices too, may be met with in the foregoing Treatiſe; but theſe few hints ſhall ſuffice.

I ſhall only mention farther the Teſtimony of Mr. *Fatio*, *R. S. S.* which we have in his Geometrical Inveſtigation of the *Line of ſhorteſt deſcent*, Printed at *London*, 1699, *pag.* 18. where we have theſe Words.

The plain and undeniable Evidence of Things, oblidges me to acknowledge Mr. Newton *not only as the Firſt, but as, by many Years, the Firſt Inventer of this* Calculus; *from whom, whether Mr.* Leibnitz, *the Second Inventer, borrowed any Thing or no, I had rather they (who have ſeen Mr.* Newton'*s Letters and other Manuſcripts) ſhould Judge, then I.*

To the ſame purpoſe is the Sentence of the *Royal Society*, annex'd to the end of the *Epiſtolical Commerce*; to which Sentence Mr. *Leibnitz* had Appeal'd, upon the occaſion of Mr. *Keil* (now *Aſtronomy Profeſſor at Oxford*) his Vindicating and aſſerting the Invention to it's proper Author. It is as follows;

" We have ſeen and conſider'd the ſeveral Letters, and Coppies
" of Letters, as well thoſe in the *Archives* of the *Royal Society*,
" as thoſe which are preſerv'd amongſt Mr. *Collins*'s Papers, be-
" tween the Years 1669, and 1677. And of theſe, thoſe which
" bore the Names of Dr. *Barrow*, Mr. *Collins*, Mr. *Oldenburgh*, and
" Mr. *Leibnitz*, we are well aſſur'd they were theirs, upon the
" Credit of ſome who were perfectly well acquainted with their
" Hand-writing. As for thoſe which bore Mr. *Gregory*'s Name, we
" are ſufficiently ſatisfied of the Genuineneſs of them, by Mr. *Col-*
" *lins*, who Tranſcrib'd ſome of them with his own Hand. Out of
" all theſe, we have taken what we thought properly belonging
" to the Buſineſs committed to us: And thoſe Excerptions which
" are now offer'd to you, together with the Letters themſelves,
" we find to be faithfully and juſtly made.

" Now from theſe Papers and Letters, the following Articles ap-
" pear plain to us.

1. " That Mr. *Leibnitz* was at *London* in the coming on of the
" Year 1673, from thence he went to *Paris* in *March*, or there-

“abouts, where He held a Correſpondence by Letters with Mr. *Collins*, by Means of Mr. *Oldenburgh*, as long as till *Sept.* 1676. And then, that He return'd to *Hannover*, by the way of *London* and *Amſterdam*. And that Mr. *Collins* did very freely Communicate to other Mathematicians, what He had receiv'd from Mr. *Newton* and Mr. *Gregory*.

2. “That Mr. *Leibnitz*, when He came to *London* the firſt time, made himſelf the Inventer of a certain Differential Method, *properly ſo called*; and tho' Dr. *Pell*, ſhew'd Him that the very ſame had been diſcover'd by Mr. *Mouton* before, yet He ſtill perſiſted to aſſert his own Right to the Invention, as well becauſe, He had (as He ſaid) found it out himſelf, having never ſeen what *Mouton* had done, as becauſe He had made ſeveral Improvements to it. Nor do we ever find any mention made of *any other Differential Method*, but this of *Mouton*'s, before what we have in his Letters dated *June* 21. 1677; that is, not till a *whole Year*, after Mr. *Newton*'s Letter of *December* 10. was ſent to *Paris* to be Communicated to Him, and *Four Years* after, Mr. *Collins* began to Communicate the ſame Letter to his Friends and Correſpondents. And 'tis plain, that the *Method of Fluxions* is evidently enough deſcribed in this Letter, to any proper Judge of theſe Matters.

3. “'Tis manifeſt from Mr. *Newton*'s Letter of *June* 13. 1676, that the *Method of Fluxions* was known to Him, *Five Years* before He wrote that Letter. And it appears by His *Analyſis of Equations having an Infinite Number of Terms*, (which Dr. *Barrow* Communicated to Mr. *Collins*, in *July* 1669.) that He had diſcover'd the Method even before *that Time*.

4. “The *Differential Method* is one and the ſame with the *Method of Fluxions*, excepting only the *Name* and the *Notation*. For Mr. *Leibnitz* calls thoſe Quantities *Differences*, which Mr. *Newton* calls *Momenta* or *Fluxions*; and He expreſſes them by the Letter *d*, which Mr. *Newton* makes no uſe of.

“And therefore we think the main Queſtion to be, *Not* who found *This* or *That* Method, but who it was that firſt of all found *the Method it ſelf*, which (in all theſe Dreſſes) is but *one*. Alſo we believe, *That thoſe Perſons who eſteem'd Mr.* Leibnitz *the firſt Inventer, knew little or nothing of the Correſpondence which paſs'd between Mr.* Collins *and Him*, nor did not know, *that Mr.* Newton *had made uſe of the ſame Method, full fifteen Years before Mr.* Leibnitz *Publiſh'd the firſt Specimens of it, in the Leipſick Acts*. From all which, *we conclude Mr.* Newton *to be the firſt Inventer of this Method, and Conſequently that Mr.* Keil, *in aſſerting it to Him, has not at all Calumniated or Injur'd Mr.* Leibnitz.

And now to put an end to this Treatiſe; I ſhall only ſay that I have herein endeavour'd to exhibit Methods that may be Accommodated to the uſe even of *Beginners*, as well as all others.

FINIS.

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